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Quantum orthogonal planes: $ISO_{q,r}(N)$ and $SO_{q,r}(N)$ – bicovariant calculi and differential geometry on quantum Minkowski space

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Abstract. We construct differential calculi on multiparametric quantum orthogonal planes in any dimension N. These calculi are bicovariant under the action of the full inhomogeneous (multiparametric) quantum group $ISO_{q,r}(N)$, and do contain dilatations. If we require bicovariance only under the quantum orthogonal group $SO_{q,r}(N)$, the calculus on the q-plane can be expressed in terms of its coordinates x^a, differentials dx^a and partial derivatives ∂_a without the need of dilatations, thus generalizing known results to the multiparametric case. Using real forms that lead to the signature $(n+1, m)$ with $m=n-1$, $n, n+1$, we find $ISO_{q,r}(n+1, m)$ and $SO_{q,r}(n+1, m)$ bicovariant calculi on the multiparametric quantum spaces. The particular case of the quantum Minkowski space $ISO_{q,r}(3,1)/SO_{q,r}(3,1)$ is treated in detail. The conjugated partial derivatives ∂_a^* can be expressed as linear combinations of the ∂_a. This allows a deformation of the phase-space where no additional operators (besides x^a and p_a) are needed.

1 Introduction

Non commutativity of spacetime at the microscopic level could provide an effective regularization of gravity, in alternative to discretization methods. It is suggestive that a non commutative structure of spacetime emerges in nonperturbative attempts to describe string theories [1].

In this paper we use the non-commuting geometry [2] of quantum groups [3,4], as defined by their differential calculi [5–12], to derive the noncommuting differential geometry of the multiparametric quantum orthogonal planes in any dimension. We then study real forms that are consistent with the differential calculus and finally specialize our treatment to the multiparametric quantum Minkowski space.

The necessary prerequisite for the work presented here has been the construction of inhomogeneous quantum groups of the orthogonal type $ISO_{q,r}(N)$ and of their corresponding bicovariant calculi. This has been achieved in past publications [13–16] via a projection from the known multiparametric orthogonal groups $SO_{q,r}(N+2)$, and has provided an R matrix formulation for the inhomogeneous case.

Other references on inhomogeneous q-groups can be found in [17,18]. For multiparametric quantum groups see [19–21].

In general, i.e. without any restrictions on the deformation parameters, inhomogeneous groups of the orthogonal type contain dilatations. It is however possible to avoid dilatations if one fixes some of the parameters (including the r parameter appearing in the off-diagonal terms of the R-matrix) equal to one, their classical value. The case $r = 1$ corresponds to a "quasi-classical" structure, for which the original braiding matrix $\hat{R}$ becomes diagonal (the corresponding deformations are then called *twistings*). In this case it is possible to construct a bicovariant calculus on $ISO_q(N)$, and consequently on q-Minkowski space [14,16].

We present here a bicovariant calculus on the full multiparametric $ISO_{q,r}(N)$ *without the restriction* $r = 1$. This calculus, however, is trivial on the $SO_{q,r}(N)$ quantum subgroup: it can really be seen as a non-trivial calculus only on the coset $Fun_{q,r}[ISO(N)/SO(N)]$, i.e. on the quantum orthogonal plane. For $r \neq 1$ this $ISO_{q,r}(N)$–bicovariant calculus on the quantum plane necessarily contains dilatations.

If we require only $SO_{q,r}(N)$ bicovariance [more precisely right covariance under $ISO_{q,r}(N)$ and left covariance only under $SO_{q,r}(N)$], the calculus can be expressed in terms of coordinates x, differentials dx and partial derivatives ∂, *without the need of dilatations*. In this case

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the q-commutations between x, dx and ∂ close by themselves, and in fact generalize to the multiparametric case the known results of [22–24]. Here these results emerge from the broader setting of the bicovariant calculus on $ISO_{q,r}(N)$. In this context we are able to explicitly relate the partial derivatives ∂ to the $ISO_{q,r}(N)$ q-Lie algebra generators.

It is natural to expect that a $*$-structure compatible with $ISO_{q,r}(N)$ and with the bicovariant differential calculus will induce a well behaved $*$-conjugation on the differential calculus on the quantum plane, acting linearly on the partial derivatives ∂_a, $(a = 1, ...N)$. The conjugations that give the real forms $ISO_{q,r}(n+1, n-1)$, $ISO_{q,r}(n, n)$ and $ISO_{q,r}(n, n+1)$ are consistent with the $ISO_q(N)$ bicovariant differential calculus. Using these conjugations one can define real coordinates X^a and hermitian partial derivative operators $P_a \sim \partial_a$, i.e. momenta. The conjugated ∂_a^* are derived from the conjugation of the $ISO_{q,r}(N)$ q-Lie algebra generators and can be simply expressed as linear combinations of the ∂_a, *without the need of introducing an extra operator* as done in [25]. The q-commutations of the momenta P with the coordinates X define a deformed phase-space that could be studied in the same spirit as in [25].

We will be concerned with the conjugation that gives the $ISO_{q,r}(n-1, n+1)$ calculus and in particular induces a differential calculus on the q-Minkowski space. To retrieve the other conjugations, both for N=even and N=odd, just take $\mathcal{D}^A{}_B = \delta^A_B$ in the formulae where $\mathcal{D}^A{}_B$ appears.

In Sect. 2 we recall briefly the structure of the $ISO_{q,r}(N)$ quantum groups. Their differential calculi are discussed in Sect. 3, and finally in Sects. 4 and 5 we present the bicovariant calculi on quantum orthogonal planes in full detail. In Appendix A we specialize our results to the four-dimensional quantum Minkowski space, and list all the relevant formulas for its non-commuting differential geometry.

2 The quantum inhomogeneous group $ISO_{q,r}(N)$

An R-matrix formulation for the quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ was obtained in [15], in terms of the $R^{AB}{}_{CD}$ matrix for the $SO_{q,r}(N+2)$ and $Sp_{q,r}(N+2)$ quantum groups. We recall here the results for $SO_{q,r}(N+2)$. The quantum inhomogeneous group $ISO_{q,r}(N)$ is freely generated by the non-commuting matrix elements $T^A{}_B$ [A=$(\circ, a, \bullet)$, with $a = 1, ...N$)] and the identity I, modulo the relations:

$$T^a{}_\circ = T^\bullet{}_b = T^\bullet{}_\circ = 0, \tag{2.1}$$

the RTT relations

$$R^{AB}{}_{EF}T^E{}_C T^F{}_D = T^B{}_F T^A{}_E R^{EF}{}_{CD}, \tag{2.2}$$

and the orthogonality relations

$$C^{BC}T^A{}_B T^D{}_C = C^{AD}, \quad C_{AC}T^A{}_B T^C{}_D = C_{BD} \tag{2.3}$$

The matrix R controls the non-commutativity of the $T^A{}_B$ elements, and its entries depend continuously on a set of parameters r, q_{AB} (q_{AB} appearing only in the diagonal part of the R matrix). For $r \to 1$, $q_{AB} \to 1$ (the "classical limit"), $R^{AB}{}_{CD} \to \delta^A_C \delta^B_D$. The quantum metric C_{AB} and its inverse C^{AB} depend only on r and are given in [4].

The co-structures of $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ are simply given by:

$$\Delta(T^A{}_B) = T^A{}_C \otimes T^C{}_B, \qquad \kappa(T^A{}_B) = C^{AC}T^D{}_C C_{DB},$$
$$\varepsilon(T^A{}_B) = \delta^A_B \,. \tag{2.4}$$

After decomposing the indices A=$(\circ, a, \bullet)$, and defining:

$$u \equiv T^\circ{}_\circ, \quad v \equiv T^\bullet{}_\bullet, \quad z \equiv T^\circ{}_\bullet, \quad x^a \equiv T^a{}_\bullet, \quad y_a \equiv T^\circ{}_a \tag{2.5}$$

the relations (2.2) and (2.3) become

$$R^{ab}{}_{ef}T^e{}_c T^f{}_d = T^b{}_f T^a{}_e R^{ef}{}_{cd} \tag{2.6}$$
$$T^a{}_b C^{bc} T^d{}_c = C^{ad} I \tag{2.7}$$
$$T^a{}_b C_{ac} T^c{}_d = C_{bd} I \tag{2.8}$$

$$T^b{}_d x^a = \frac{r}{q_{d\bullet}} R^{ab}{}_{ef} x^e T^f{}_d \tag{2.9}$$
$$P_A^{ab}{}_{cd} x^c x^d = 0 \tag{2.10}$$
$$T^b{}_d v = \frac{q_{b\bullet}}{q_{d\bullet}} v T^b{}_d \tag{2.11}$$
$$x^b v = q_{b\bullet} v x^b \tag{2.12}$$
$$uv = vu = I \tag{2.13}$$
$$u x^b = q_{b\bullet} x^b u \tag{2.14}$$
$$u T^b{}_d = \frac{q_{b\bullet}}{q_{d\bullet}} T^b{}_d u \tag{2.15}$$

$$y_b = -r^{\frac{N}{2}} T^a{}_b C_{ac} x^c u \tag{2.16}$$

$$z = -\frac{1}{(r^{-\frac{N}{2}} + r^{\frac{N}{2}-2})} x^b C_{ba} x^a u \tag{2.17}$$

where $q_{a\bullet}$ are N complex parameters related by $q_{a\bullet} = r^2/q_{a'\bullet}$, with $a' = N+1-a$. The matrix P_A in Eq. (2.10) is the q-antisymmetrizer for the B, C, D q-groups given by (cf. (B.4)):

$$P_A^{ab}{}_{cd} = -\frac{1}{r+r^{-1}}(\hat{R}^{ab}{}_{cd} - r\delta^a_c\delta^b_d + \frac{r-r^{-1}}{r^{N-2}+1}C^{ab}C_{cd}). \tag{2.18}$$

The last two relations (2.16), (2.17) are constraints, showing that the $T^A{}_B$ matrix elements in Eq. (2.2) are really a *redundant* set. This redundance is necessary if we want to express the q-commuations of the $ISO_{q,r}(N)$ basic group elements as $RTT = TTR$ (i.e. if we want an R-matrix formulation). We can take as independent generators the elements

$$T^a{}_b, x^a, v, u \equiv v^{-1} \text{ and the identity } I \qquad (a = 1, ...N) \tag{2.19}$$

The co-structures on the $ISO_{q,r}(N)$ generators can be read from (2.4) after decomposing the indices $A = \circ, a, \bullet$:

$$\Delta(T^a{}_b) = T^a{}_c \otimes T^c{}_b \ ,$$
$$\Delta(x^a) = T^a{}_c \otimes x^c + x^a \otimes v \ , \qquad (2.20)$$
$$\Delta(v) = v \otimes v \ , \quad \Delta(u) = u \otimes u \ , \qquad (2.21)$$

$$\kappa(T^a{}_b) = C^{ac} T^d{}_c C_{db} \ , \qquad (2.22)$$
$$\kappa(x^a) = -\kappa(T^a{}_c) x^c u \ , \quad \kappa(v) = u \ , \quad \kappa(u) = v, \qquad (2.23)$$

$$\varepsilon(T^a{}_b) = \delta^a_b \ , \quad \varepsilon(x^a) = 0 \ , \quad \varepsilon(u) = \varepsilon(v) = \varepsilon(I) = 1 \ . \qquad (2.24)$$

In the commutative limit $q \to 1, r \to 1$ we recover the algebra of functions on $ISO(N)$ (plus the dilatation v that can be set to the identity).

Note 2.1: as shown in [15], the quantum group $ISO_{q,r}(N)$ can be derived as the quotient

$$\frac{SO_{q,r}(N+2)}{H} \ , \qquad (2.25)$$

where H is the Hopf ideal in $SO_{q,r}(N+2)$ of all sums of monomials containing at least an element of the kind $T^a{}_\circ, T^\bullet{}_b, T^\bullet{}_\circ$. The Hopf structure of the groups in the numerators of (2.25) is naturally inherited by the quotient groups [27].

We denote by P the canonical projection

$$P \ : \ S_{q,r}(N+2) \longrightarrow S_{q,r}(N+2)/H \qquad (2.26)$$

It is a Hopf algebra epimorphism because $H = Ker(P)$ is a Hopf ideal. Then any element of $S_{q,r}(N+2)/H$ is of the form $P(a)$ and

$$P(a) + P(b) \equiv P(a+b) \ ; \quad P(a)P(b) \equiv P(ab) \ ;$$
$$\mu P(a) \equiv P(\mu a), \quad \mu \in \mathbf{C} \qquad (2.27)$$
$$\Delta(P(a)) \equiv (P \otimes P)\Delta_{N+2}(a) \ ; \quad \varepsilon(P(a)) \equiv \varepsilon_{N+2}(a) \ ;$$
$$\kappa(P(a)) \equiv P(\kappa_{N+2}(a)) \qquad (2.28)$$

where we indicate by Δ_{N+2}, ε_{N+2} and κ_{N+2} the co-structures of $SO_{q,r}(N+2)$. Equations (2.6)–(2.17) have been obtained in [15] by taking the P projection of the RTT and CTT relations of $S_{q,r}(N+2)$, with the notation $u \equiv P(T^\circ{}_\circ)$, $v \equiv P(T^\bullet{}_\bullet)$, $z \equiv P(T^\circ{}_\bullet)$, $x^a \equiv P(T^a{}_\bullet)$, $y_a \equiv P(T^\circ{}_a)$, $T^a{}_b \equiv P(T^a{}_b)$; $I \equiv P(I)$; $0 \equiv P(0)$, cf. (2.5).

Note 2.2: From the commutations (2.14) - (2.15) we see that one can set $u = I$ only when $q_{a\bullet} = 1$ for all a. From $q_{a\bullet} = r^2/q_{a'\bullet}$ this implies also $r = 1$.

Note 2.3: Equations (2.10) are just the multiparametric orthogonal quantum plane commutations. Choosing as free indices $({}^{\bullet b}_{\bullet\bullet})$ in (2.2) yields $zx^b = q_{b\bullet} x^b z$ and therefore the (squared) length element $L = x^a C_{ab} x^b$ commutes with the x elements. Similarly we find $LT^a{}_d = (q_{d\bullet}/r)^2 T^a{}_d L$ and $Lu = r^{-2} uL, \ Lv = r^2 vL$.

Note 2.4: Two conjugations (i.e. algebra antihomomorphisms, coalgebra homomorphisms and involutions, satisfying $\kappa(\kappa(T^*)^*) = T$) exist on $ISO_{q,r}(N)$, inherited from the corresponding ones on $SO_{q,r}(N+2)$ [15,16]. We recall here their action on the group generators $T^A{}_B$ and the corresponding restrictions on the parameters:

- trivially as $T^* = T$; corresponds to the real forms $ISO_{q,r}(n,n;\boldsymbol{R})$ and $ISO_{q,r}(n,n+1;\boldsymbol{R})$. Compatibility with the RTT relations (2.2) requires $|q_{ab}| = |q_{a\bullet}| = |r| = 1$.
- Only for $N = 2n$ even: $(T^A{}_B)^* = \mathcal{D}^A_C T^C{}_D \mathcal{D}^D_B$, $\mathcal{D}$ being the matrix that exchanges the index n with the index $n+1$; extends to the inhomogeneous multiparametric case the conjugation proposed by the authors of [26] for $SO_q(2n, \mathbf{C})$, and corresponds to the real form $ISO_{q,r}(n+1, n-1; \boldsymbol{R})$.

Explicitly: $(T^a{}_b)^* = \mathcal{D}^a_c T^c{}_d \mathcal{D}^d_b, (x^a)^* = \mathcal{D}^a{}_b x^b, u^* = u, v^* = v, z^* = z$. Compatibility with the RTT relations (2.2) requires:

$$(\bar{R})_{n\leftrightarrow n+1} = R^{-1}, \quad \text{i.e.} \quad \mathcal{D}_1\mathcal{D}_2 R_{12} \mathcal{D}_1\mathcal{D}_2 = \overline{R}^{-1}_{12} \qquad (2.29)$$

which implies $|r| = 1$; $|q_{a\bullet}| = 1$ for $a \neq n, n+1$; $|q_{ab}| = 1$ for a and b both different from n or $n+1$; $q_{ab}/r \in \boldsymbol{R}$ when at least one of the indices a, b is equal to n or $n+1$; $q_{a\bullet}/r \in \boldsymbol{R}$ for $a = n$ or $a = n+1$. Compatibility with the CTT relations (2.3) is ensured by $\mathcal{DCD}$ and $\bar{C} = C^T$ (due to $|r| = 1$).

In particular, the quantum Poincaré group $ISO_{q,r}(3,1;\boldsymbol{R})$ is obtained by setting $|q_{1\bullet}| = |r| = 1$, $q_{2\bullet}/r \in \boldsymbol{R}$, $q_{12}/r \in \boldsymbol{R}$.

According to Note 2.2, a *dilatation-free* q-Poincaré group is found after the further restrictions $q_{1\bullet} = q_{2\bullet} = r = 1$. The only free parameter remaining is then $q_{12} \in \boldsymbol{R}$.

3 Bicovariant calculus on simple q-groups

The bicovariant differential calculus on the q-groups of the A, B, C, D series can be formulated in terms of the corresponding R-matrix, or equivalently in terms of the $L^\pm$ functionals defined by:

$$L^{\pm A}{}_B(T^C{}_D) = (R^\pm)^{AC}{}_{BD} \ , \quad L^{\pm A}{}_B(I) = \delta^A_B \qquad (3.1)$$

with [1]

$$(R^+)^{AC}{}_{BD} \equiv R^{CA}{}_{DB} \ , \quad (R^-)^{AC}{}_{BD} \equiv (R^{-1})^{AC}{}_{BD} \ . \qquad (3.2)$$

To extend the definition (3.1) to the whole Hopf algebra $\mathcal{A}$ we set

$$L^{\pm A}{}_B(ab) = L^{\pm A}{}_C(a) L^{\pm C}{}_B(b) \quad \forall a, b \in \mathcal{A} \ . \qquad (3.3)$$

These functionals generate the Hopf algebra $\mathcal{A}'$ paired to $\mathcal{A}$, with $\Delta'(L^{\pm A}{}_B) = L^{\pm A}{}_C \otimes L^{\pm C}{}_B$, $\varepsilon'(L^{\pm A}{}_B) = \delta^A_B$ and $\kappa'(L^\pm) = (L^\pm)^{-1}$.

[1] for the B, C, D series. For the q-groups of the A series there is more freedom in choosing R^+ and R^-, see for ex. [13]

We briefly recall how to construct a bicovariant calculus. The general procedure can be found in [7,12], or, in the notations we adopt here, in [10]. It realizes the axiomatic construction of [5].

The functionals

$$f_{A_1}{}^{A_2B_1}{}_{B_2} \equiv \kappa(L^{+B_1}{}_{A_1})L^{-A_2}{}_{B_2}. \tag{3.4}$$

and the elements of $\mathcal{A}$:

$$M^{B_1}{}_{B_2A_1}{}^{A_2} \equiv T^{B_1}{}_{A_1}\kappa(T^{A_2}{}_{B_2}). \tag{3.5}$$

satisfy the following relations, called *bicovariant bimodule conditions*, where for simplicity we use the adjoint indices $i, j, k, ...$ with ${}^i = {}_A{}^B,\ {}_i = {}^A{}_B$:

$$\Delta'(f^i{}_j) = f^i{}_k \otimes f^k{}_j \quad ; \quad \varepsilon'(f^i{}_j) = \delta^i_j\ , \tag{3.6}$$

$$\Delta(M_j{}^i) = M_j{}^l \otimes M_l{}^i \quad ; \quad \varepsilon(M_j{}^i) = \delta^i_j\ , \tag{3.7}$$

$$M_i{}^j(a * f^i{}_k) = (f^j{}_i * a)M_k{}^i \tag{3.8}$$

The star product between a functional on $\mathcal{A}$ and an element of $\mathcal{A}$ is defined as:

$$\chi * a \equiv (id\otimes\chi)\Delta(a), \quad a*\chi \equiv (\chi\otimes id)\Delta(a), \quad a \in \mathcal{A},\ \chi \in \mathcal{A}' \tag{3.9}$$

Relation (3.8) is easily checked for $a = T^A{}_B$ since in this case it is implied by the RTT relations; it holds for a generic a because of property (3.6).

The space of **quantum one-forms** is defined as a left $\mathcal{A}$-module Γ freely generated by the *symbols* $\omega_{A_1}{}^{A_2}$:

$$\Gamma \equiv \{a^{A_1}{}_{A_2}\omega_{A_1}{}^{A_2}\}\ , \quad a^{A_1}{}_{A_2} \in \mathcal{A} \tag{3.10}$$

Theorem 3.1 (due to Woronowicz, see last ref. in [5]): because of the properties (3.6), Γ becomes a bimodule over $\mathcal{A}$ with the following right multiplication:

$$\omega_{A_1}{}^{A_2}a = (f_{A_1}{}^{A_2B_1}{}_{B_2} * a)\ \omega_{B_1}{}^{B_2}, \tag{3.11}$$

in particular:

$$\omega_{A_1}{}^{A_2}T^R{}_S = (R^{-1})^{TB_1}{}_{CA_1}(R^{-1})^{A_2C}{}_{B_2S}T^R{}_T\omega_{B_1}{}^{B_2} \tag{3.12}$$

Moreover, because of properties (3.7) and (3.8), we can define a left and a right action of $\mathcal{A}$ on Γ:

$$\Delta_L : \Gamma \to \mathcal{A}\otimes\Gamma \ ;$$
$$\Delta_L(a\omega_{A_1}{}^{A_2}b) \equiv \Delta(a)\,(I\otimes\omega_{A_1}{}^{A_2})\,\Delta(b)\ , \tag{3.13}$$
$$\Delta_R : \Gamma \to \Gamma\otimes\mathcal{A} \ ;$$
$$\Delta_R(a\omega_{A_1}{}^{A_2}b) \equiv \Delta(a)\,(\omega_{B_1}{}^{B_2}\otimes M^{B_1}{}_{B_2A_1}{}^{A_2})\,\Delta(b)\ . \tag{3.14}$$

These actions commute, i.e. $(id\otimes\Delta_R)\Delta_L = (\Delta_L\otimes id)\Delta_R$, and give a bicovariant bimodule structure to Γ. [Property (3.8) is a sufficient and necessary condition for $\Delta_R(\rho b) = \Delta_R(\rho)\Delta(b)$].

The elements $M_j{}^i$ can be used to build a right-invariant basis of Γ. Indeed the η^i defined by

$$\eta^i \equiv \kappa^{-1}(M_j{}^i)\omega^j \tag{3.15}$$

are right invariant: $\Delta_R(\eta^i) = \eta^i\otimes I$ [use $\kappa^{-1}(a_2)a_1 = \varepsilon(a)$]; moreover every element of Γ can be written as $\rho = \eta^ib_i$ or $a_i\eta^i$ where b_i and a_i are uniquely determined. It can be shown that the functionals $f^i{}_j$ satisfy:

$$\eta^ib = (b * f^i{}_j)\eta^j \tag{3.16}$$

$$a\eta^i = \eta^j[a * (f^i{}_j \circ \kappa)], \tag{3.17}$$

where $a * f \equiv (f\otimes id)\Delta(a)$.

The **exterior derivative** $d\ :\ \mathcal{A} \longrightarrow \Gamma$ can be defined via the element $\tau \equiv \sum_A \omega_A{}^A \in \Gamma$. This element is easily shown to be left and right-invariant:

$$\Delta_L(\tau) = I\otimes\tau \quad ; \quad \Delta_R(\tau) = \tau\otimes I \tag{3.18}$$

and defines the derivative d by

$$da = \frac{1}{r - r^{-1}}[\tau a - a\tau]. \tag{3.19}$$

The factor $\frac{1}{r-r^{-1}}$ is necessary for a correct classical limit $r \to 1$. It is immediate to prove the Leibniz rule

$$d(ab) = (da)b + a(db), \quad \forall a, b \in \mathcal{A}\ . \tag{3.20}$$

Two other expressions for the derivative are given by:

$$da = (\chi^{A_1}{}_{A_2} * a)\ \omega_{A_1}{}^{A_2}\ , \tag{3.21}$$

$$da = -\eta_{A_1}{}^{A_2}(a * \kappa'(\chi^{A_1}{}_{A_2})) \tag{3.22}$$

where the linearly independent elements

$$\chi^A{}_B = \frac{1}{r - r^{-1}}[f_C{}^{CA}{}_B - \delta^A_B\varepsilon] \tag{3.23}$$

are the tangent vectors such that the left-invariant vector fields $\chi^A{}_B*$ are dual to the left-invariant one-forms $\omega_{A_1}{}^{A_2}$, and similarly the right-invariant vector fields $*\kappa'(\chi^{A_1}{}_{A_2})$ are dual to the right-invariant one forms $-\eta_{A_1}{}^{A_2}$. The equivalence of (3.19) and (3.21) can be shown by using the rule (3.11) for τa in the right-hand side of (3.19). Expression (3.22) is related to (3.21) via $\chi_i * a = (a * \chi_j)\kappa^{-1}(M_i{}^j)$, Eqs. (3.15), (3.17) and $\kappa'(\chi_i) = -\chi_j\kappa'(f^j{}_i)$.
Using (3.21) we can compute the exterior derivative on the basis elements $T^A{}_B$:

$$d\,T^A{}_B = \frac{1}{r - r^{-1}}[(R^{-1})^{CR}{}_{ET}(R^{-1})^{TE}{}_{SB}T^A{}_C - \delta^R_ST^A{}_B]\ \omega_R{}^S \tag{3.24}$$

Every element ρ of Γ, which by definition is written in a unique way as $\rho = a^{A_1}{}_{A_2}\omega_{A_1}{}^{A_2}$, can also be written as

$$\rho = \sum_k a_kdb_k \tag{3.25}$$

for some a_k, b_k belonging to $\mathcal{A}$. This can be proven directly by inverting the relation (3.24).

Due to the bi-invariance of τ the derivative operator d is compatible with the actions Δ_L and Δ_R:

$$\begin{aligned} \Delta_L(adb) &= \Delta(a)(id\otimes d)\Delta(b)\ , \\ \Delta_R(adb) &= \Delta(a)(d\otimes id)\Delta(b)\ , \end{aligned} \tag{3.26}$$

these two properties express the fact that d commutes with the left and right action of the quantum group, as in the classical case.

Remark : The properties (3.20), (3.25) and (3.26) of the exterior derivative (3.21) realize the axioms of a first-order bicovariant differential calculus [5].

The **tensor product** between elements $\rho, \rho' \in \Gamma$ is defined to have the properties $\rho a \otimes \rho' = \rho \otimes a\rho'$, $a(\rho\otimes\rho') = (a\rho)\otimes\rho'$ and $(\rho\otimes\rho')a = \rho\otimes(\rho' a)$. Left and right actions on $\Gamma\otimes\Gamma$ are defined by:

$$\begin{aligned} &\Delta_L(\rho\otimes\rho') \equiv \rho_1\rho_1'\otimes\rho_2\otimes\rho_2', \\ &\quad\Delta_L : \Gamma\otimes\Gamma\to\mathcal{A}\otimes\Gamma\otimes\Gamma \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\Delta_R(\rho\otimes\rho') \equiv \rho_1\otimes\rho_1'\otimes\rho_2\rho_2', \\ &\quad\Delta_R : \Gamma\otimes\Gamma\to\Gamma\otimes\Gamma\otimes\mathcal{A} \end{aligned} \tag{3.28}$$

where ρ_1, ρ_2, etc., are defined by:

$$\begin{aligned} \Delta_L(\rho) &= \rho_1\otimes\rho_2, \quad \rho_1\in\mathcal{A},\ \rho_2\in\Gamma\ ; \\ \Delta_R(\rho) &= \rho_1\otimes\rho_2,\ \rho_1\in\Gamma,\ \rho_2\in\mathcal{A}\ . \end{aligned}$$

The extension to $\Gamma^{\otimes n}$ is straightforward.

The **exterior product** of one-forms is consistently defined as:

$$\omega_{A_1}{}^{A_2}\wedge\omega_{D_1}{}^{D_2} \equiv \omega_{A_1}{}^{A_2}\otimes\omega_{D_1}{}^{D_2} - \Lambda^{A_2}_{A_1}{}^{D_2}_{D_1}|^{C_1}_{C_2}{}^{B_1}_{B_2}\omega_{C_1}{}^{C_2}\otimes\omega_{B_1}{}^{B_2} \tag{3.29}$$

where the Λ tensor is given by:

$$\begin{aligned} \Lambda^{A_2}_{A_1}{}^{D_2}_{D_1}|^{C_1}_{C_2}{}^{B_1}_{B_2} &\equiv f_{A_1}{}^{A_2B_1}{}_{B_2}(M^{C_1}_{C_2D_1}{}^{D_2}) \\ &= d^{F_2}d^{-1}_{C_2}R^{F_2B_1}{}_{C_2G_1}(R^{-1})^{C_1G_1}{}_{E_1A_1} \\ &\quad\times(R^{-1})^{A_2E_1}{}_{G_2D_1}R^{G_2D_2}{}_{B_2F_2} \end{aligned} \tag{3.30}$$

d^A being the entries of the diagonal matrix $D^A_B \equiv C^{AC}C_{BC}$. From the formula (3.29) we can find the q-commutations (generalizing the classical anticommutations) of products of one-forms ω in terms of a "flip" operator (see the second ref. in [11]):

$$\omega^i\wedge\omega^j = -Z^{ij}{}_{kl}\omega^k\wedge\omega^l \tag{3.31}$$

Using the exterior product we can define the **exterior differential on** Γ :

$$d\ :\ \Gamma\to\Gamma\wedge\Gamma\quad ;\quad d(a_k db_k) = da_k\wedge db_k \tag{3.32}$$

which can easily be extended to $\Gamma^{\wedge n}$ ($d:\Gamma^{\wedge n}\to\Gamma^{\wedge(n+1)}$, $\Gamma^{\wedge n}$ being defined as in the classical case but with the quantum permutation (braid) operator Λ [5]). The definition (3.32) is equivalent to the following :

$$d\theta = \frac{1}{r-r^{-1}}[\tau\wedge\theta - (-1)^k\theta\wedge\tau], \tag{3.33}$$

where $\theta\in\Gamma^{\wedge k}$. The exterior differential has the following properties:

$$\begin{aligned} d(\theta\wedge\theta') &= d\theta\wedge\theta' + (-1)^k\theta\wedge d\theta'\quad ; \\ d(d\theta) &= 0\ , \end{aligned} \tag{3.34}$$

$$\begin{aligned} \Delta_L(\theta d\theta') &= \Delta_L(\theta)(id\otimes d)\Delta_L(\theta')\quad ; \\ \Delta_R(\theta d\theta') &= \Delta_R(\theta)(d\otimes id)\Delta_R(\theta'), \end{aligned} \tag{3.35}$$

where $\theta\in\Gamma^{\wedge k}$, $\theta'\in\Gamma^{\wedge n}$.

The ***q* -Cartan-Maurer equations** are found by using (3.33) in computing $d\omega_{C_1}{}^{C_2}$:

$$\begin{aligned} d\omega_{C_1}{}^{C_2} &= \frac{1}{r-r^{-1}}(\omega_B{}^B\wedge\omega_{C_1}{}^{C_2} + \omega_{C_1}{}^{C_2}\wedge\omega_B{}^B) \\ &\equiv -\frac{1}{2}C^{A_1}_{A_2}{}^{B_1}_{B_2}|^{C_2}_{C_1}\ \omega_{A_1}{}^{A_2}\wedge\omega_{B_1}{}^{B_2} \end{aligned} \tag{3.36}$$

with:

$$C^{A_1}_{A_2}{}^{B_1}_{B_2}|^{C_2}_{C_1} = -\frac{2}{(r-r^{-1})}[Z_B{}^B{}_{C_1}{}^{C_2}|^{A_1}_{A_2}{}^{B_1}_{B_2} + \delta^{A_1}_{C_1}\delta^{C_2}_{A_2}\delta^{B_1}_{B_2}] \tag{3.37}$$

To derive this formula we have used the flip operator Z on $\omega_B{}^B\wedge\omega_{C_1}{}^{C_2}$.

Finally, we recall that the χ operators close on the ***q*-Lie algebra** :

$$\chi_i\chi_j - \Lambda^{kl}{}_{ij}\chi_k\chi_l = \boldsymbol{C}_{ij}{}^k\chi_k \tag{3.38}$$

where the q-structure constants are given by

$$\begin{aligned} \boldsymbol{C}_{jk}{}^i &= \chi_k(M_j{}^i)\quad \text{explicitly :} \\ \boldsymbol{C}^{A_1}_{A_2}{}^{B_1}_{B_2}|^{C_2}_{C_1} &= \frac{1}{r-r^{-1}}[-\delta^{B_1}_{B_2}\delta^{A_1}_{C_1}\delta^{C_2}_{A_2} \\ &\quad + \Lambda^B_B{}^{C_2}_{C_1}|^{A_1}_{A_2}{}^{B_1}_{B_2}]. \end{aligned} \tag{3.39}$$

The C structure constants appearing in the Cartan-Maurer equations are in general related to the $\boldsymbol{C}$ constants of the q-Lie algebra [10]:

$$\boldsymbol{C}_{jk}{}^i = \frac{1}{2}[C_{jk}{}^i - \Lambda^{rs}{}_{jk}C_{rs}{}^i]\ . \tag{3.40}$$

Using the definitions (3.23) and (3.4) it is not difficult to find the co-structures on the functionals χ and f:

$$\begin{aligned} &\Delta'(\chi_i) = \chi_j\otimes f^j{}_i + \varepsilon\otimes\chi_i\ ;\quad \Delta'(f^i{}_j) = f^i{}_k\otimes f^k{}_j\ , \\ &\varepsilon'(\chi_i) = 0\ ;\quad \varepsilon'(f^i{}_j) = \delta^i_j\ , \\ &\kappa'(\chi_i) = -\chi_j\kappa'(f^j{}_i)\ ;\quad \kappa'(f^k{}_j)f^j{}_i = \delta^k_i\varepsilon = f^k{}_j\kappa'(f^j{}_i)\ . \end{aligned} \tag{3.41}$$

Note that in the $r,q \to 1$ limit $f^i{}_j \to \delta^i_j \varepsilon$, i.e. $f^i{}_j$ becomes proportional to the identity functional and formula (3.11), becomes trivial, e.g. $\omega^i a = a\omega^i$ [use $\varepsilon * a = a$].

The $*$-conjugation on $\mathcal{A}$ is canonically extended to a conjugation on the Hopf algebra $\mathcal{A}'$ generated by the $L^{\pm}$ functional and paired to $\mathcal{A}$. The relation is

$$\phi^*(a) \equiv \overline{\phi(\kappa^{-1}(a^*))} \tag{3.42}$$

where the overline denotes complex conjugation. Explicitly, on the χ functionals it reads [2] [16]

$$(\chi^A{}_B)^* = -r^{-N-1}\chi^C{}_D \mathcal{D}^F{}_B \mathcal{D}^A{}_G R^{EG}{}_{FC} D^D{}_E \text{ for } SO_{q,r}(n+2,n;\boldsymbol{R})\ ;\quad 2n+2 = N+2 \tag{3.43}$$

with $D^D{}_E \equiv C^{DF}C_{EF}$. Since the conjugation is a linear operation on the q-Lie algebra, it can be extended via (3.21) to the differential calculus as well [5,18]. The unique antilinear involution $*$ on Γ satisfies:

$$(a\rho)^* = \rho^* a^*,\ (\rho a)^* = a^*\rho^*, (da)^* = d(a^*); \tag{3.44}$$
$$\Delta_L(\rho^*) = \Delta_L(\rho)^*,\quad \Delta_R(\rho^*) = \Delta_R(\rho)^* \tag{3.45}$$

where $(a \otimes \rho)^* = a^* \otimes \rho^*$ and $(\rho \otimes a)^* = \rho^* \otimes a^*$. It easily extends to $\Gamma^{\wedge n}$, for ex. $(d\theta)^* = d\theta^*$ etc. Inverting formulae (3.24) one can also find the induced conjugation on the left-invariant one-forms [16]. The explicit relation between the $*$-structures on the χ and on the ω can be given using the duality $\langle \omega_A{}^B, \chi^C{}_D\rangle = \delta^C_A \delta^B_D$ between left-invariant vector fields and left-invariant one-forms [18]:

$$\langle \omega_A{}^{B\,*}, \chi^C{}_D\rangle = -\overline{\langle \omega_A{}^B, \chi^C{}_D{}^*\rangle}\ . \tag{3.46}$$

4 Bicovariant differential calculus on $ISO_{q,r}(N)$

The existence of this calculus is simply due to the existence of a subset of the functionals (3.4), vanishing on the Hopf ideal H (see Sect. 2) , and M elements that satisfy the bicovariant bimodule conditions (3.6)-(3.8). These are:

$$f_\bullet{}^{\circ\bullet}{}_\circ\ ,\quad f_\bullet{}^{a\bullet}{}_\circ\ ,\quad f_\bullet{}^{\bullet\bullet}{}_\circ\ ,\quad f_\bullet{}^{a\bullet}{}_b\ ,\quad f_\bullet{}^{\bullet\bullet}{}_b\ ,\quad f_\bullet{}^{\bullet\bullet}{}_\bullet \tag{4.1}$$

$$\begin{aligned}
&P(M^\bullet{}_{\circ\bullet}{}^\circ) = v^2\\
&P(M^\bullet{}_{\circ\bullet}{}^d) = 0\\
&P(M^\bullet{}_{\circ\bullet}{}^\bullet) = 0\\
&P(M^\bullet{}_{b\bullet}{}^\circ) = vr^{-\frac{N}{2}}x^e C_{eb}\\
&P(M^\bullet{}_{b\bullet}{}^d) = v\kappa(T^d{}_b)\\
&P(M^\bullet{}_{b\bullet}{}^\bullet) = 0\\
&P(M^\bullet{}_{\bullet\bullet}{}^\circ) = -\frac{1}{r^N(r^{\frac{N}{2}} + r^{-\frac{N}{2}+2})}x^e C_{ef} x^f\\
&P(M^\bullet{}_{\bullet\bullet}{}^d) = v\kappa(x^d)\\
&P(M^\bullet{}_{\bullet\bullet}{}^\bullet) = I
\end{aligned} \tag{4.2}$$

[2] We correct here a misprint of [16] where the factor r^{-N+1} instead of r^{-N-1} appears. Notice that there we have used the opposite convention $\phi^*(a) \equiv \overline{\phi(\kappa(a^*))}$ instead of (3.42).

Notice that only the couples of indices $(\bullet\circ)$, $(\bullet b)$ and $(\bullet\bullet)$ appear in (4.1)-(4.2): these are therefore the only indices involved in the projected differential calculus on $ISO_{q,r}(N)$.

Theorem 4.1: the functionals $f^i{}_j$ in (4.1) annihilate the Hopf ideal H.

Proof: one first checks directly that the functionals (4.1) vanish on the generators $\mathcal{T}$ of the ideal H, i.e. on $\mathcal{T} = T^a{}_\circ, T^\bullet{}_b, T^\bullet{}_\circ$. This extends to any element of $H = Ker(P)$, because of the property (3.6). Q.E.D.

These functionals are then well defined on the quotient $ISO_{q,r}(N) = SO_{q,r}(N+2)/Ker(H)$, in the sense that the "projected" functionals

$$\bar{f} : ISO_{q,r}(N) \to \boldsymbol{C},\quad \bar{f}(P(a)) \equiv f(a)\ ,\quad \forall a \in SO_{q,r}(N+2) \tag{4.3}$$

are well defined. Indeed if $P(a) = P(b)$, then $f(a) = f(b)$ because $f(Ker(P)) = 0$. This holds for any functional f vanishing on $Ker(P)$.

The product fg of two generic functionals vanishing on $KerP$ also vanishes on $KerP$, because $KerP$ is a co-ideal (see [15]): $fg(KerP) = (f \otimes g)\Delta_{N+2}(KerP) = 0$. Therefore $\overline{fg}$ is well defined on $ISO_{q,r}(N)$; moreover, [use (2.28)] $\overline{fg}[P(a)] \equiv fg(a) = (\bar{f} \otimes \bar{g})\Delta(P(a)) \equiv \bar{f}\bar{g}[P(a)]$, and the product of $\bar{f}$ and $\bar{g}$ involves the coproduct Δ of $ISO_{q,r}(N)$.

There is a natural way to introduce a coproduct on the $\bar{f}$'s :

$$\begin{aligned}\Delta\bar{f}[P(a) \otimes P(b)] &\equiv \bar{f}[P(a)P(b)] = \bar{f}[P(ab)]\\ &= f(ab) = \Delta_{N+2} f(a \otimes b)\ .\end{aligned} \tag{4.4}$$

It is also easy to show that

$$\begin{aligned}\Delta(\bar{f}^i{}_j) &= \bar{f}^i{}_k \otimes \bar{f}^k{}_j \quad \text{i.e.}\\ \bar{f}^i{}_j[P(a)P(b)] &= \bar{f}^i{}_k[P(a)]\bar{f}^k{}_j[P(b)]\end{aligned} \tag{4.5}$$

with i,j,k running over the restricted set of indices $\bullet b, \bullet\bullet, \bullet\circ$. Indeed due to the vanishing of some f's (a consequence of upper and lower triangularity of L^+ and L^- respectively), formulae (3.41) and (3.6) involve only the $f^i{}_j$ listed in (4.1). Then

$$\begin{aligned}\bar{f}^i{}_j[P(a)P(b)] &= \bar{f}^i{}_j[P(ab)] = f^i{}_j(ab) = f^i{}_k(a)f^k{}_j(b)\\ &= \bar{f}^i{}_k[P(a)]\bar{f}^k{}_j[P(b)]\end{aligned} \tag{4.6}$$

and (4.5) is proved.

With abuse of notations we will simply write f instead of $\bar{f}$. Then the f in (4.1) will be seen as functionals on $ISO_{q,r}(N)$.

Theorem 4.2: the left $\mathcal{A}$-module ($\mathcal{A} = ISO_{q,r}(N)$) Γ freely generated by $\omega^i \equiv \omega_\bullet{}^a, \omega_\bullet{}^\bullet$ and $\omega_\bullet{}^\circ$ is a bicovariant bimodule over $ISO_{q,r}(N)$ with right multiplication:

$$\omega^i a = (f^i{}_j * a)\omega^j, \quad a \in ISO_{q,r}(N) \tag{4.7}$$

where the $f^i{}_j$ are given in (4.1), the $*$-product is computed with the co-product Δ of $ISO_{q,r}(N)$, and the left and right actions of $ISO_{q,r}(N)$ on Γ are given by

$$\Delta_L(a_i\omega^i) \equiv \Delta(a_i)I \otimes \omega^i \tag{4.8}$$
$$\Delta_R(a_i\omega^i) \equiv \Delta(a_i)\omega^j \otimes M_j{}^i \tag{4.9}$$

where the $M_j{}^i$ are given in (4.2).

Proof: by showing that the functionals f and the elements M listed in (4.1) and (4.2) satisfy the properties (3.6)-(3.8) (cf. Theorem 3.1). It is straightforward to verify directly that the elements M in (4.2) do satisfy the properties (3.7). We have already shown that the functionals f in (4.1) satisfy (3.6) ($\varepsilon(f^i{}_j) = \delta^i_j$ obviously also holds for this subset).

Consider now the last property (3.8). We know that it holds for $SO_{q,r}(N+2)$. Take the free indices j and k as $\bullet b$, $\bullet\bullet$ and $\bullet\circ$, and apply the projection P on both members of the equation. It is an easy matter to show that only the f's in (4.1) and the M's in (4.2) enter the sums: this is due to the vanishing of some $P(M)$ and to some f's. We still have to prove that the $*$ product in (3.8) can be computed via the coproduct Δ in $ISO_{q,r}(N)$. Consider the projection of property (3.8), written symbolically as:

$$P[M(f \otimes id)\Delta_{N+2}(a)] = P[(id \otimes f)\Delta_{N+2}(a)M] \ . \tag{4.10}$$

Now apply the definition (4.3) and the first of (2.28) to rewrite (4.10) as

$$P(M)(\bar{f} \otimes id)\Delta(P(a)) = (id \otimes \bar{f})\Delta(P(a))P(M) \ . \tag{4.11}$$

This projected equation then *becomes* property (3.8) for the $ISO_{q,r}(N)$ functionals f and adjoint elements M, with the correct coproduct Δ of $ISO_{q,r}(N)$. Q.E.D.

To simplify notations, we write the composite indices as follows:

$$_\bullet{}^a \to {}^a, \ _\bullet{}^\bullet \to {}^\bullet, \ _\bullet{}^\circ \to {}^\circ; \quad {}^\bullet{}_a \to {}_a, \ {}^\bullet{}_\bullet \to {}_\bullet, \ {}^\bullet{}_\circ \to {}_\circ \tag{4.12}$$

Similarly we'll write q_b instead of $q_{b\bullet}$.

Using the general formula (4.7) we can deduce the ω, T commutations for $ISO_{q,r}(N)$:

$$\omega^b T^c{}_d = \frac{q_f}{r}(R^{-1})^{bf}{}_{ed}T^c{}_f\omega^e \tag{4.13}$$
$$\omega^b x^c = \frac{q_b}{r^2}x^c\omega^b + \lambda r^{\frac{N}{2}-1}q_d C^{bd}T^c{}_d\omega^\circ \tag{4.14}$$
$$\omega^b u = \frac{r^2}{q_b}u\ \omega^b \tag{4.15}$$
$$\omega^b v = \frac{q_b}{r^2}v\ \omega^b \tag{4.16}$$
$$\omega^\bullet T^c{}_d = T^c{}_d\omega^\bullet \tag{4.17}$$
$$\omega^\bullet x^c = \frac{1}{r^2}x^c\omega^\bullet - \lambda\frac{q_b}{r}T^c{}_b\omega^b \tag{4.18}$$
$$\omega^\bullet u = r^2 u\omega^\bullet \tag{4.19}$$
$$\omega^\bullet v = r^{-2}v\omega^\bullet \tag{4.20}$$
$$\omega^\circ T^c{}_d = q_d^2 r^{-2}T^c{}_d\omega^\circ \tag{4.21}$$
$$\omega^\circ x^c = x^c\omega^\circ \tag{4.22}$$
$$\omega^\circ u = u\omega^\circ \tag{4.23}$$
$$\omega^\circ v = v\omega^\circ \tag{4.24}$$

where $\lambda \equiv r - r^{-1}$.

Note 4.1: u commutes with all ω 's only if $r = q_a = 1$ (cf. *Note 2.2*). This means that $u = I$ is consistent with the differential calculus on $ISO_{q_{ab};r=q_a=1}(N)$.

The 1-form $\tau \equiv \omega^\bullet \equiv \omega_\bullet{}^\bullet$ is bi-invariant, as one can check by using (4.8)-(4.9). Then an exterior derivative on $ISO_{q,r}(N)$ can be defined as in Eq. (3.19), and satisfies the Leibniz rule. The alternative expression $da = (\chi_i * a)\omega^i$ (cf. (3.21)) continues to hold, where

$$\chi_b = \frac{1}{r - r^{-1}}f^\bullet{}_b$$
$$\chi_\circ = \frac{1}{r - r^{-1}}f^\bullet{}_\circ$$
$$\chi_\bullet = \frac{1}{r - r^{-1}}[f^\bullet{}_\bullet - \varepsilon] \tag{4.25}$$

are the left-invariant vectors dual to the left-invariant 1-forms ω^b, $\omega^\bullet$ and ω°. As a consequence of (4.5) their coproduct is given by

$$\Delta(\chi_b) = \chi_\bullet \otimes f^\bullet{}_b + \chi_c \otimes f^c{}_b + \varepsilon \otimes \chi_b \tag{4.26}$$
$$\Delta(\chi_\bullet) = \chi_\bullet \otimes f^\bullet{}_\bullet + \varepsilon \otimes \chi_\bullet \tag{4.27}$$
$$\Delta(\chi_\circ) = \chi_\circ \otimes f^\circ{}_\circ + \chi_\bullet \otimes f^\bullet{}_\circ + \chi_c \otimes f^c{}_\circ + \varepsilon \otimes \chi_\circ \tag{4.28}$$

The exterior derivative on the generators of $ISO_{q,r}(N)$ reads:

$$dT^c{}_d = 0 \tag{4.29}$$
$$dx^c = -q_b r^{-1}T^c{}_b\omega^b - r^{-1}x^c\omega^\bullet \tag{4.30}$$
$$du = ru\omega^\bullet \tag{4.31}$$
$$dv = -r^{-1}v\omega^\bullet \tag{4.32}$$
$$dz = -q_b r^{-1}y_b\omega^b - r(1 - r^N)u\omega^\circ - r^{-1}z\omega^\bullet \tag{4.33}$$

where we have included the exterior derivative on z for convenience. Note that the calculus is trivial on the $SO_{q,r}(N)$ subgroup of $ISO_{q,r}(N)$, as is evident from (4.29). Thus effectively we are discussing a bicovariant calculus on the orthogonal q-plane generated by the coordinates x^a and the "dilatations" u, v.

Every element ρ of Γ can be written as $\rho = \sum_k a_k db_k$ for some a_k, b_k belonging to $ISO_{q,r}(N)$. Indeed inverting the relations (4.30)-(4.33) yields:

$$\omega^a = -\frac{r}{q_{a\bullet}}\kappa(T^a{}_c)[dx^c - x^c u dv] \tag{4.34}$$

$$\omega^\bullet = -rudv = r^{-1}vdu \tag{4.35}$$

$$\omega^\circ = -\frac{vdz + r^{-N}zdv + r^{-\frac{N}{2}}C_{ab}x^a dx^b}{r(1-r^N)} \tag{4.36}$$

Finally, the two properties (3.26) hold also for $ISO_{q,r}(N)$, because of the bi-invariance of $\tau = \omega^\bullet$. Thus all the axioms for a bicovariant first order differential calculus on $ISO_{q,r}(N)$ are satisfied.

The exterior product of left-invariant one-forms is defined as

$$\omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}{}_{kl}\omega^k \otimes \omega^l \tag{4.37}$$

where

$$\Lambda^{ij}{}_{kl} = f^i{}_l(M_k{}^j) \tag{4.38}$$

This Λ tensor can in fact be obtained from the one of $SO_{q,r}(N+2)$ by restricting its indices to the subset $\bullet b, \bullet\bullet, \bullet\circ$. This is true because when $i, l = \bullet b, \bullet\bullet, \bullet\circ$ we have $f^i{}_l(KerP) = 0$ so that $f^i{}_l$ is well defined on $ISO_{q,r}(N)$, and we can write $f^i{}_l(M_k{}^j) = \bar{f}^i{}_l[P(M_k{}^j)]$ (see discussion after Theorem 4.1). The non-vanishing components of Λ read:

$$\begin{array}{ll}
\Lambda^{ad}{}_{cb} = \frac{q_a}{q_c}r^{-1}R^{ad}{}_{bc} & \Lambda^{\bullet\circ}{}_{cb} = -\frac{r^{-\frac{N}{2}-1}}{q_c}\lambda C_{bc} \\
\Lambda^{\bullet d}{}_{c\bullet} = r^{-2}\delta^d_c & \Lambda^{a\circ}{}_{c\circ} = r^{-1}\lambda\delta^a_c \\
\Lambda^{\circ d}{}_{c\circ} = (\frac{r}{q_c})^2\delta^d_c & \Lambda^{a\bullet}{}_{\bullet b} = \delta^a_b \\
\Lambda^{\bullet d}{}_{\bullet b} = r^{-1}\lambda\delta^d_b & \Lambda^{a\circ}{}_{\circ b} = r^{-4}(q_a)^2\delta^a_b \\
\Lambda^{\bullet\bullet}{}_{\bullet\bullet} = 1 & \Lambda^{ad}{}_{\bullet\circ} = -q_a r^{-\frac{N}{2}-1}\lambda C^{da} \\
\Lambda^{\bullet\circ}{}_{\bullet\circ} = \lambda r^{-1}(1-r^{-N}) & \Lambda^{\circ\bullet}{}_{\bullet\circ} = 1 \\
\Lambda^{\bullet\circ}{}_{\circ\bullet} = r^{-4} & \Lambda^{\circ\circ}{}_{\circ\circ} = 1
\end{array}$$

From (4.37) it is not difficult to deduce the commutations between the ω's:

$$\frac{1}{q_c}P_S^{ab}{}_{cd}\ \omega^d \wedge \omega^c = 0 \tag{4.39}$$

$$\omega^a \wedge \omega^\bullet = -r^2\omega^\bullet \wedge \omega^a \tag{4.40}$$

$$\omega^a \wedge \omega^\circ = -r^{-4}(q_a)^2\omega^\circ \wedge \omega^a \tag{4.41}$$

$$\omega^\bullet \wedge \omega^\bullet = \omega^\circ \wedge \omega^\circ = 0 \tag{4.42}$$

$$\omega^\bullet \wedge \omega^\circ = -r^{-4}\omega^\circ \wedge \omega^\bullet + \frac{\lambda r^{-\frac{N}{2}-1}}{q_a(1-r^{-N})}C_{ba}\ \omega^a \wedge \omega^b \tag{4.43}$$

where P_S is the q-symmetrizer given in Appendix B. Notice that the dimension of the space of 2-forms generated by $\omega^a \wedge \omega^b$ is larger than in the commutative case since P_S projects into an $N(N+1)/2 - 1$ (and not into an $N(N+1)/2$) dimensional space. This is not surprising since the exterior algebra of homogeneous orthogonal quantum groups is known to be larger than its classical counterpart.

The exterior differential on $\Gamma^{\wedge n}$ can be defined as in Sect. 4 (Eq. (3.33)), and satisfies all the properties (3.34), (3.35). As for $SO_{q,r}(N+2)$ the last two hold because of the bi-invariance of $\tau \equiv \omega^\bullet$.

The Cartan-Maurer equations

$$d\omega^i = \frac{1}{r-r^{-1}}(\tau \wedge \omega^i + \omega^i \wedge \tau) \tag{4.44}$$

can be explicitly found after use of the commutations (4.39)- (4.43):

$$d\omega^a = r^{-1}\omega^a \wedge \omega^\bullet \tag{4.45}$$

$$d\omega^\bullet = 0 \tag{4.46}$$

$$d\omega^\circ = -r(1+r^2)\omega^\bullet \wedge \omega^\circ + \frac{r^3}{r^{\frac{N}{2}} - r^{-\frac{N}{2}}}\frac{C_{ba}}{q_a}\omega^a \wedge \omega^b \tag{4.47}$$

The nonvanishing structure constants $\boldsymbol{C}$ (appearing in the q-Lie algebra , see below), given by $\boldsymbol{C}_{jk}{}^i = \chi_k(M_j{}^i)$, read:

$$\begin{array}{ll}
\boldsymbol{C}_{ab}{}^\circ = -q_a^{-1}r^{-\frac{N}{2}-1}C_{ba} & \boldsymbol{C}_{a\bullet}{}^c = -r^{-1}\delta^c_a \\
\boldsymbol{C}_{\bullet b}{}^c = r^{-1}\delta^c_b & \boldsymbol{C}_{\circ\bullet}{}^\circ = -r^{-3}(1+r^2) \\
\boldsymbol{C}_{\bullet\circ}{}^\circ = r^{-1}(1-r^{-N}) &
\end{array}$$

These structure constants can be obtained from those of $SO_{q,r}(N+2)$ by specializing indices, for essentially the same reason as for the Λ components.

Using the values of the Λ and $\boldsymbol{C}$ tensors given above, we can explicitly write the q-Lie algebra of translations and dilatations on $ISO_{q,r}(N)$ as:

$$\chi_\circ\chi_b - q_b^2 r^{-4}\ \chi_b\chi_\circ = 0 \tag{4.48}$$

$$\chi_c\chi_\bullet - r^{-2}\chi_\bullet\chi_c = -r^{-1}\chi_c \tag{4.49}$$

$$\chi_\circ\chi_\bullet - r^{-4}\chi_\bullet\chi_\circ = \frac{-(1+r^2)}{r^3}\chi_\circ \tag{4.50}$$

$$q_a P_A^{ab}{}_{cd}\chi_b\chi_a = 0 \tag{4.51}$$

A combination of the above relations yields:

$$\chi_\circ + \lambda\ \chi_\bullet\chi_\circ = \lambda\ \frac{-q_a r^{-\frac{N}{2}}}{r^{-2}+r^{-N}}\ \chi_a C^{ba}\chi_b \tag{4.52}$$

Notice the similar structure of Eqs. (2.17) and (4.52).

The *-conjugation on the χ functionals and on the one-forms ω can be deduced from (3.43) [use $(q_f)^{-1}\mathcal{D}^f{}_b = \bar{q}_b\mathcal{D}^f{}_b$]

$$(\chi_b)^* = -r^{-N}\mathcal{D}^f{}_b\frac{1}{q_f}D^d{}_f\chi_d = -r^{-N}\bar{q}_b\mathcal{D}^f{}_b D^d{}_f\chi_d = -r^{-N}\bar{q}_b D^f{}_b\mathcal{D}^d{}_f\chi_d \tag{4.53}$$

$$(\chi_\bullet)^* = -\chi_\bullet \tag{4.54}$$

$$(\chi_\circ)^* = -r^{-2N-2}\chi_\circ \tag{4.55}$$

whereas the conjugation on the ω one-forms can be deduced from (3.46) and (4.53)- (4.55) or directly from their expression in terms of dx, du, dv differentials (4.34)-(4.36) remembering that $(da)^* = d(a^*)$:

$$(\omega^a)^* = \bar{q}_a{}^{-1}r^N(D^{-1})^a{}_b\mathcal{D}^b{}_c = \bar{q}_a{}^{-1}r^N\mathcal{D}^a{}_b(D^{-1})^b{}_c\omega^c \tag{4.56}$$

$$(\omega^\bullet)^* = \omega^\bullet \tag{4.57}$$

$$(\omega^\circ)^* = r^{2N+2}\omega^\circ\ . \tag{4.58}$$

5 Calculus on the multiparametric orthogonal quantum plane: coordinates, differentials and partial derivatives

5.1 $ISO_{q,r}(N)$ bicovariant calculus

In this section we concentrate on the orthogonal quantum plane

$$M \equiv Fun_{q,r}\left(\frac{ISO(N)}{SO(N)}\right) , \qquad (5.1)$$

i.e. the $ISO_{q,r}(N)$ subalgebra generated by the coordinates x^a and the dilatations u, v.

We study the action of the exterior differential d on M and the corresponding space Γ_M of 1-forms. Γ_M is the subbimodule of Γ formed by all the elements adb or $(da')b'$ where a, b, a', b' are polynomials in x^a, u and v [of course $adb = d(ab) - (da)b$].

We will see that a generic element ρ of Γ_M cannot be generated, as a left module, only by the differentials dx, dv, i.e. it cannot be written as $\rho = a_i dx^i + adv$. We need also to introduce the differential dz (or equivalently $dL \equiv d(x^a C_{ab} x^b)$). Thus the basis of differentials is given by dx^a, dv, dz and corresponds to the intrinsic basis of *independent* one-forms $\omega^a, \omega^\bullet$ and ω°. Note that du can be expressed in terms of dv since $du = -u(dv)u = -r^2 u^2 dv = -r^{-2}(dv)u^2$ [see (5.70) below].

Commutations

The commutations between the coordinates x^a, u and v have been given in Sect. 2. The commutations between coordinates and differentials are found by expressing the differentials in terms of the one-forms ω as in (4.30)-(4.33), and using then the x, u, v commutations with the ω's given in (4.13)-(4.24). The resulting q-commutations between x and dx are found to be:

$$(r^{-2}P_S - P_A)(x \otimes dx) = (P_S + P_A)(dx \otimes x) \qquad (5.2)$$

where the projectors P_S and P_A are defined in (B.4), and we have used the tensor notation $A^{ab}{}_{cd} x^c dx^d \equiv A(x \otimes dx)$ etc. The remaining commutations are given in formulae (5.67)–(5.78) in Table 1.

Let us consider the above formula, giving the x, dx commutations. If we multiply it by P_0 we find $0 = 0$. Thus from this equation we have no information on $P_0(x \otimes dx)$. Applying instead the projectors P_S and P_A yields

$$P_S(x \otimes dx) = r^2 P_S(dx \otimes x) \ ; \quad P_A(x \otimes dx) = -P_A(dx \otimes x) \qquad (5.3)$$

which does not allow to express $x^a dx^b$ only in terms of linear combinations of $(dx)x$ since no linear combination of P_S and P_A is invertible. The space of 1-forms has therefore one more dimension than his classical analogue because we are missing a condition involving the one dimensional projector $P_{0\,ef}^{ab} = Q_N(r) C^{ab} C_{\ ef}$, see (B.4).

However, if we consider the 1-form $dL \equiv d(x^e C_{ef} x^f)$ – an exterior derivative of *polynomials* in the basic elements – we can write the commutations between the x and dx elements as follows:

$$\begin{aligned} dx \otimes x &= -(P_S + P_A + P_0) x \otimes dx + (P_S + P_A) d(x \otimes x) \\ &\quad + P_0 d(x \otimes x) \\ &= P_S dx \otimes x + P_A dx \otimes x - P_0 x \otimes dx + P_0 d(x \otimes x) \\ &= (r^{-2} P_S - P_A - P_0) x \otimes dx + P_0 d(x \otimes x) \qquad (5.4) \end{aligned}$$

where we have used the Leibniz rule, the commutations (5.3) and $P_S + P_A + P_0 = I$. Equivalently we have

$$\begin{aligned} dx \otimes x &= (r^{-2} P_S - P_A - P_0) x \otimes dx \\ &\quad - C \frac{r^{\frac{N}{2}-2}(1 - r^2)}{1 - r^N} (vdz + zdv) \qquad (5.5) \end{aligned}$$

involving the dv and dz differentials.

The presence of dz can also be explained within the general theory by recalling that Γ is a free right module [see paragraph following (3.15)]. A basis of right invariant 1-forms is given by (3.15) and we explicitly have:

$$\eta^a = -r^{-1} dx^a\, u = -r^{-1} dT^a{}_\bullet\, \kappa(T^\bullet{}_\bullet) \qquad (5.6)$$

$$\eta^\bullet = -r^{-1} dv\, u = -r^{-1} dT^\bullet{}_\bullet\, \kappa(T^\bullet{}_\bullet) \qquad (5.7)$$

$$\begin{aligned} \eta^\circ &= \frac{r^{\frac{N}{2}-1}}{(1 - r^N)(1 + r^{N-2})} \\ &\quad \times [dx^e C_{ef} x^f - r^{N-2} x^e C_{ef} dx^f] u^2 \qquad (5.8) \\ &= \frac{-r^{N-1}}{r^N - 1} [dz\, u + dy_b\, \kappa(x^b) + du\, \kappa(z)] \\ &= \frac{-r^{N-1}}{r^N - 1} dT^\circ{}_B\, \kappa(T^B{}_\bullet) \qquad (5.9) \end{aligned}$$

To derive the expression for η° use: $y_b = -r^{-\frac{N}{2}} u x^e C_{ef} T^f{}_b$; $dy_b\, \kappa(x^b) = r^{-\frac{N}{2}} du\, xxu + r^{-\frac{N}{2}} dx\, xu^2$; $dz = d(\frac{-r^{-\frac{N}{2}}}{1+r^{2-N}} uxx)$ $= \frac{-r^{-\frac{N}{2}}}{1+r^{2-N}} (du\, xx + dx\, xu + xdx\, u)$; $\kappa(z) = \kappa(T^\circ{}_\bullet) = r^{-N} z$; $uxx = r^2 xxu$; $udx\, x = dx\, xu$, where $xx \equiv L \equiv x^e C_{ef} x^f$.

The 1-forms (5.7)-(5.8) in Γ do not contain any $T^a{}_b$ element and therefore belong to Γ_M as well; they are linearly independent and freely generate Γ_M as a right module because they freely generate the full Γ as a right module. The extra 1-form η° (or dz) is therefore a natural consequence of the right module structure of Γ.

In summary: either dL or dz or η° are necessary in order to close the commutation algebra between coordinates and differentials. Thus the commutations involving z and dz appear in Table 1.

We have seen that $dv\, u$; $dx^a\, u$ and η° freely generate Γ_M as a right module; recalling that Γ is also a free left module, we have the :

Proposition 5.1 The M-bimodule Γ_M, as a left module (or as a right module), is freely generated by the differentials dx, dL (or dz) and dv. *Proof:* to show that $a_i dx^i + adL + a_\bullet dv = 0 \Rightarrow a_i = 0, a = 0, a_\bullet = 0$ express dx^i, dL, dv in terms of $\omega^a, \omega^\circ \omega^\bullet$, see (4.30)-(4.33), and recall that Γ is a free left module.

Note 5.1 From (5.67), (5.68) and the commutations of L with x and u we have $x^c dL = dL\, x^c$, $udL = dL\, u$ and $vdL = dL\, v$. These relations and (5.66) show that inside Γ_M there is the smaller bimodule generated by the differentials dx^a and dL.

We now examine the space of 2-forms. By simply applying the exterior derivative d to the relations (5.66)-(5.78) we deduce the commutations between the differentials given in Table 1. As with the ω^a's in Eq. (4.39), the relations in (5.79) are not sufficient to order the differentials dx^a.

ISO$_{q,r}$(N) **- coactions**

All the relations we have been deriving have many symmetries properties because they are covariant under the actions on M and Γ of the full $ISO_{q,r}(N)$ q-group. In fact we have the following three $ISO_{q,r}(N)$ actions:

1) the coproduct of $ISO_{q,r}(N)$ can be seen as a left-coaction $\Delta :\ M \to ISO_{q,r}(N) \otimes M$:
$$\Delta x^a = T^a{}_b \otimes x^b + x^a \otimes v \tag{5.10}$$
2) the left coaction $\Delta_L : \Gamma \to ISO_{q,r}(N) \otimes \Gamma$, when restricted to Γ_M gives
$$\Delta_L|_{\Gamma_M} : \Gamma_M \to ISO_{q,r}(N) \otimes \Gamma_M \tag{5.11}$$
and defines a left coaction of $ISO_{q,r}(N)$ on Γ_M compatible with the bimodule structure of Γ_M and the exterior differential: $\Delta_L|_{\Gamma_M}(adb) = \Delta(a)(id \otimes d)\Delta(b)$.
3) the right coaction $\Delta_R : \Gamma \to \Gamma \otimes ISO_{q,r}(N)$ does not become a right coaction of $ISO_{q,r}(N)$ on Γ_M; however we have
$$\Delta_R|_{\Gamma_M} : \Gamma_M \to \Gamma \otimes M \subset \Gamma \otimes ISO_{q,r}(N) \tag{5.12}$$
this map is obviously well defined and satisfies $\Delta_R|_{\Gamma_M}(adb) = \Delta(a)(d \otimes id)\Delta(b)$ $\forall a, b \in M$ since $M \subset ISO_{q,r}(N)$.

We call this calculus $ISO_{q,r}(N)$-bicovariant because $\Delta_L|_{\Gamma_M}$ and $\Delta_R|_{\Gamma_M}$ are compatible with the bimodule structure of Γ_M and with the exterior differential.

Partial derivatives

The tangent vectors χ in (4.25) and the corresponding vector fields $\chi*$ have "flat" indices. To compare $\chi*$ with partial derivative operators with "curved" indices, we need to define the operators $\overleftarrow{\partial}$:
$$\overleftarrow{\partial}_c(a) \equiv -\frac{r}{q_b}(\chi_b * a)\kappa(T^b{}_c) - \frac{r^{-\frac{N}{2}}}{r(1-r^N)}\,(\chi_\circ * a)C_{bc}x^b, \tag{5.13}$$
$$\overleftarrow{\partial}_\bullet(a) \equiv \frac{r}{q_b}(\chi_b * a)\kappa(T^b{}_c)x^c u - r(\chi_\bullet * a)u - \frac{r^{-N}}{r(1-r^N)}\,(\chi_\circ * a)z \tag{5.14}$$
$$\overleftarrow{\partial}_\circ(a) \equiv -\frac{1}{r(1-r^N)}(\chi_\circ * a)v \tag{5.15}$$
so that
$$d\,a = \overleftarrow{\partial}_c(a)\, dx^c + \overleftarrow{\partial}_\bullet(a)dv + \overleftarrow{\partial}_\circ(a)dz \equiv \overleftarrow{\partial}_C(a)dx^C \tag{5.16}$$
$[C = (\circ, c, \bullet),\ dx^C = (dz, dx^c, dv)]$ which is equation $da = (\chi_c * a)\omega^c + (\chi_\bullet * a)\omega^\bullet + (\chi_\circ * a)\omega^\circ$ in "curved" indices. The action of $\overleftarrow{\partial}_C$ on the coordinates $x^C = (z, x^c, v)$ is given by
$$\overleftarrow{\partial}_C(x^A) = \delta^A_C I\ , \tag{5.17}$$
From the Leibniz rule $d(ab) = (da)b + a(db)$, using (5.16) and the fact that $dx^C = (dz, dx^c, dv)$ is a basis for 1-forms, we find for example:
$$\overleftarrow{\partial}_c(ax^b) = a\delta^b_c + \overleftarrow{\partial}_d(a)(r^{-2}P_S - P_A - P_0)^{db}{}_{ec}x^e - (1-r^2)\overleftarrow{\partial}_\bullet(a)\delta^b_c v \tag{5.18}$$
The tangent vector fields χ_c* of this paper and the partial derivatives $\overleftarrow{\partial}$ are derivative operators that act "from the right to the left" as can be seen from the deformed Leibniz rule (5.18). This explains the inverted arrow on $\overleftarrow{\partial}$.

Equation (5.18) gives the $\overleftarrow{\partial}_c, x^b$ commutations. The rest of the $\overleftarrow{\partial}_C, x^B$ commutations reads:
$$\overleftarrow{\partial}_\bullet(ax^b) = q_b^{-1}\overleftarrow{\partial}_\bullet(a)x^b - C^{cb}\frac{r^{\frac{N}{2}-2}(1-r^2)}{(1-r^N)}\ \overleftarrow{\partial}_c(a)z \tag{5.19}$$
$$\overleftarrow{\partial}_\circ(ax^b) = q_b\overleftarrow{\partial}_\circ(a)x^b - C^{cb}\frac{r^{\frac{N}{2}-2}(1-r^2)}{(1-r^N)}\ \overleftarrow{\partial}_c(a)v \tag{5.20}$$
$$\overleftarrow{\partial}_c(av) = r^{-2}q_c\overleftarrow{\partial}_c(a)v \tag{5.21}$$
$$\overleftarrow{\partial}_\bullet(av) = r^{-2}\overleftarrow{\partial}_\bullet(a)v + a \tag{5.22}$$
$$\overleftarrow{\partial}_\circ(av) = \overleftarrow{\partial}_\circ(a)v \tag{5.23}$$
$$\overleftarrow{\partial}_c(az) = q_c^{-1}\overleftarrow{\partial}_c(a)z \tag{5.24}$$
$$\overleftarrow{\partial}_\bullet(az) = r^{-2}\overleftarrow{\partial}_\bullet(a)z \tag{5.25}$$
$$\overleftarrow{\partial}_\circ(az) = r^2\overleftarrow{\partial}_\circ(a)z + a + (r^{-2}-1)\overleftarrow{\partial}_\bullet(a)v \tag{5.26}$$
We can also define derivative operators acting from the left to the right, as in [23], using the antipode κ which is antimultiplicative [one can also use (3.22)]. For a generic quantum group the vectors $-\kappa'^{-1}(\chi_i) \equiv -\chi_i \circ \kappa^{-1}$ act from the left and we also have
$$d\,a = (\chi_i * a)\omega^i = \omega^i(-\kappa'^{-1}(\chi_i) * a) \tag{5.27}$$
as is seen from $\kappa'(\chi_i) = -\chi_j\kappa'(f^j{}_i)$ and $\kappa'^{-1}(f^k{}_j)f^i{}_k = \delta^i_j$ [third line of (3.41)].

We then define the partial derivatives ∂_C so that
$$d\,a = dx^C\ \partial_C(a)\ . \tag{5.28}$$
Again the action of ∂_C on the coordinates is
$$\partial_C(x^A) = \delta^A_C I\ , \tag{5.29}$$
The ∂_C, x^B commutations are given in Table 1.

5.2 $SO_{q,r}(N)$ bicovariant calculus

Commutations

Since the $P_A{}^{ab}{}_{cd}x^cx^d = 0$ commutation relations allow for an ordering of the coordinates (moreover the Poincaré series of the polynomials on the quantum orthogonal plane is the same as the classical one), it is tempting to impose extra conditions on the differential algebra of the q-Minkowski plane, so that the space of 1-forms has the same dimension as in the classical case. We require that the commutation relations between x and dx close on the algebra generated by x and dx:

$$dx^a x^b = \alpha^{ab}{}_{ef} x^e dx^f \tag{5.30}$$

where α is an unknown matrix whose entries are complex numbers. Any such matrix can be expanded as $\alpha = aP_S + bP_A + cP_0$ with $a, b, c = const.$ From (5.3) we have $\alpha = r^{-2}P_S - P_A + cP_0$; therefore condition (5.30) is equivalent to

$$P_0(dx \otimes x) = cP_0(x \otimes dx) \tag{5.31}$$

and supplements Eqs. (5.3). Taking its exterior derivative leads to a supplementary condition on the dx, dx products (for $c \neq -1$):

$$P_0(dx \wedge dx) = 0 \ . \tag{5.32}$$

From (5.79) and (5.32) it follows that $dx \wedge dx = (P_S + P_A + P_0)(dx \wedge dx) = P_A(dx \wedge dx)$, or [see the definition of P_A in (B.4)] :

$$dx \wedge dx = -r\hat{R}\ dx \wedge dx \ . \tag{5.33}$$

which allows the ordering of dx, dx products.

Using (5.4), (5.31) and (B.3), we find

$$\begin{aligned} dx \otimes x &= (r^{-2}P_S - P_A)(x \otimes dx) + P_0(dx \otimes x) \\ &= (r^{-2}P_S - P_A)(x \otimes dx) + cP_0(x \otimes dx) \\ &= (r^{-2}P_S - P_A + r^{N-2}P_0)(x \otimes dx) \\ &\quad +(c - r^{N-2})P_0(x \otimes dx) \\ &= r^{-1}\hat{R}^{-1}(x \otimes dx) \\ &\quad +(c - r^{N-2})P_0(x \otimes dx) \ . \end{aligned} \tag{5.34}$$

The consistency of the commutation relations (5.33) and (5.34) with the associativity condition on the triple $dx^i\, dx^j\, x^k$ fixes $c = r^{N-2}$ i.e.:

$$P_0(dx \otimes x) = r^{N-2}P_0(x \otimes dx) \ ; \tag{5.35}$$

the x, dx commutations (5.34) then become:

$$x \otimes dx = r\hat{R}(dx \otimes x) \tag{5.36}$$

and reproduce (in the uniparametric case) the known x, dx commutations of the quantum orthogonal plane [24].

Coactions

This calculus is no more covariant under the $ISO_{q,r}(N)$ action,

$$x^a \longrightarrow T^a{}_b \otimes x^b + x^a \otimes v \ , u \longrightarrow u \otimes u,\ v \longrightarrow v \otimes v \tag{5.37}$$

but we are left with covariance under the $SO_{q,r}(N)$ action

$$x^a \longrightarrow T^a{}_b \otimes x^b \ . \tag{5.38}$$

In other words, $\delta_L : \Gamma'_M \to SO_{q,r} \otimes \Gamma'_M$ defined by $\delta_L(adb) = \delta(c)(id \otimes d)\delta(b)$ with $\delta(x^a) = T^a{}_b \otimes x^b$ is a left coaction of $SO_{q,r}(N)$ on the bimodule Γ'_M where Γ'_M is Γ_M with the extra condition (5.31) [cf. (5.11)]. Similarly, the map $\delta_R(adb) = \delta(a)(d \otimes id)\delta(b)$ is well defined [cf. (5.12)].

Left covariance under (5.37) is broken only by (5.31). Indeed, while relations (5.3) are left and right $ISO_{q,r}(N)$-covariant, the extra condition (5.31) is *not* left $ISO_{q,r}(N)$-covariant : $\Delta_L[P_0(dx \otimes x) - cP_0(x \otimes dx)] \neq 0, \forall c$. It is right $ISO_{q,r}(N)$-covariant, $\Delta_R[P_0(dx \otimes x) - cP_0(x \otimes dx)] = 0$, only for $c = r^{N-2}$, as can be seen using $T^b{}_d dx^a = d(T^b{}_d x^a) = \frac{r}{q_d} R^{ab}{}_{ef} dx^e\, T^f{}_d$ and (B.6). Therefore the choice $c = r^{N-2}$ preserves the right coaction Δ_R.

Note 5.2 We can reformulate the quotient procedure $\Gamma_M \to \Gamma'_M$ in a more abstact setting by considering that Γ_M is a subbimodule of the *bicovariant* bimodule Γ. In (5.8) we have expressed the $x^e C_{ef} dx^f \leftrightarrow dx^e C_{ef} x^f$ commutation via the right invariant 1-form η°. A condition on Γ (and therefore on Γ_M) that preserves *right* $ISO_{q,r}(N)$ *covariance*, i.e. compatible with Δ_R [as given in (4.9)], is: η° linearly dependent from the remaining right invariant 1-forms $dv\, u$ and $dx^a\, u$. It is easily seen that since η° is quadratic in the basis elements x^a the only possible linear condition is $\eta^\circ = 0$, and this gives exactly (5.35). The M-bimodule Γ'_M is therefore generated by the differentials dx^b and dv. Since left $ISO_{q,r}(N)$ covariance is broken (whereas right $ISO_{q,r}(N)$ covariance is preserved), the relation between the left invariant 1-forms is *nonlinear*. Explicitly we have

$$\omega^\circ = -\frac{q_a}{r^2} v y_a\ \omega^a + \frac{r^{\frac{N}{2}-2}}{r^N + r^2} C_{ab} x^a x^b \omega^\bullet \tag{5.39}$$

[express dz in terms of dx^i,dv in (4.36) and use the expansion of dx^b and dv on ω^a and $\omega^\bullet$ as given in (4.30),(4.32)].

Partial derivatives

The relevant $\overleftarrow{\partial}$ operators reduce to $\overleftarrow{\partial}_c$, $\overleftarrow{\partial}_\bullet$, and the $\overleftarrow{\partial}_C, x^b$, commutations become:

$$\begin{aligned} \overleftarrow{\partial}_c(ax^b) &= a\delta^b_c + \overleftarrow{\partial}_d(a) r^{-1}(\hat{R}^{-1})^{db}{}_{ec} x^e \\ &\quad -(1 - r^2)\overleftarrow{\partial}_\bullet \delta^b_c v \end{aligned} \tag{5.40}$$

$$\overleftarrow{\partial}_\bullet(ax^b) = q_b^{-1} \overleftarrow{\partial}_\bullet(a) x^b \tag{5.41}$$

while the $\overleftarrow{\partial}_C, v$ commutations are unchanged. Note the dilatation operator $\overleftarrow{\partial}_\bullet$ appearing on the right-hand side of (5.18) or (5.40). The ∂_C, x^B commutations with $C = (c, \bullet)$, $B = (b, \bullet)$ are collected in Table 2.

From $d^2(a) = 0 = d(\overleftarrow{\partial}_C(a) dx^C) = \overleftarrow{\partial}_B(\overleftarrow{\partial}_C(a)) dx^B \wedge dx^C$ and the q-commutations of the differentials (5.79)-(5.85) one finds the commutations between the partial derivatives:

$$(P_A)^{ab}{}_{cd}\overleftarrow{\partial}_a\overleftarrow{\partial}_b = 0 \tag{5.42}$$
$$\overleftarrow{\partial}_b\overleftarrow{\partial}_\bullet - \frac{q_b}{r^2}\overleftarrow{\partial}_\bullet\overleftarrow{\partial}_b = 0 \tag{5.43}$$
$$\overleftarrow{\partial}_b\overleftarrow{\partial}_\circ - q_b\overleftarrow{\partial}_\circ\overleftarrow{\partial}_b = 0 \tag{5.44}$$
$$\overleftarrow{\partial}_\bullet\overleftarrow{\partial}_\circ - \overleftarrow{\partial}_\circ\overleftarrow{\partial}_\bullet = 0 \tag{5.45}$$

Similarly the ∂_A, ∂_B commutations are given in Table 2.

We now give an explicit relation between the ∂_c, $\partial_\bullet$ and the q-Lie algebra generators χ_c, $\chi_\bullet$ (a similar expression holds also for the $\overleftarrow{\partial}$ derivatives). Recalling (3.22) we have:

$$d\,a = -\eta^C(a * \kappa'(\chi_C)) \tag{5.46}$$

where here $C = (c, \bullet)$ because we have set $\eta^\circ = 0$. Putting together (5.46) and (5.6),(5.7), the relations $d\,a = dx^C\ \partial_C(a)$ give, $\forall a \in ISO_{q,r}(N)$:

$$\partial_c(a) = r^{-1}u(a * \kappa'(\chi_c)) \quad , \qquad \partial_\bullet(a) = r^{-1}u(a * \kappa'(\chi_\bullet)) \ . \tag{5.47}$$

The commutations between the partial derivatives given in Table 2 were obtained from $d^2 = 0$, but can be also derived via (5.47) and the q-Lie algebra (4.48)-(4.51).

Similarly we can introduce the right invariant vector fields

$$h_C \equiv h_{\kappa'(\chi_C)} \equiv [\kappa'(\chi_C) \otimes id]\Delta \tag{5.48}$$

and use their Leibniz rule [which follows from $\Delta(\kappa'(\chi_C)) = \kappa'(\chi_C) \otimes \varepsilon + \kappa'(f^D{}_C) \otimes \kappa'(\chi_D)$]:

$$h_C(ab) = h_C(a)\,b + \kappa'(f^D{}_C)(a_1)\ a_2 h_D(b) \tag{5.49}$$

to rederive the ∂, x, u commutations. For example we have $h_a x^b = rv\delta^b_a + (r/q_b)R^{eb}{}_{ac}x^c h_e + r\lambda h_\bullet$ that together with $\partial_C = r^{-1}uh_C$ (cf. (5.47)) gives

$$\partial_a x^b = \delta^i_j[I + (r^2-1)v\partial_\bullet] + rR^{eb}{}_{ac}x^c\partial_e \ .$$

Conjugation

The commutations in Table 2 are consistent under the conjugation (already defined for x^a and dx^a)

$$(x^a)^* = \mathcal{D}^a{}_b x^b,\ (dx^a)^* = \mathcal{D}^a{}_b dx^b,\ (\partial_a)^* = -r^N d_b^{-1}\mathcal{D}^b{}_a\partial_b \tag{5.50}$$
$$v^* = v,\ (dv)^* = dv,\ (\partial_\bullet)^* = u - \partial_\bullet \tag{5.51}$$

where the entries d_a have been defined after (3.30). This can be proved directly by taking the *-conjugates of the relations in Table 2, and by using the identity (2.29) and:

$$\bar{C} = C^T;\ [Q_N(r)]^* = Q_N(r); \tag{5.52}$$

$$d^c d_h^{-1} R^{cg}{}_{ha}(R^{-1})^{ea}{}_{cd} = \delta^e_h\delta^g_d;\quad R^{ab}{}_{cd}d^a d^b = R^{ab}{}_{cd}d^c d^d$$
$$d^c R^{cg}{}_{hc} = r^{N-1}\delta^g_h;\quad R^{ab}{}_{cd}d_a^{-1}d_b^{-1} = R^{ab}{}_{cd}d_c^{-1}d_d^{-1} \tag{5.53}$$

$$\overline{q_a} = \frac{1}{q_a} \text{ for } a \neq n, n+1,\quad \overline{q_n} = \frac{1}{q_{n+1}} \tag{5.54}$$

We now *derive* the conjugation on the partial derivatives from the differential calculus on $ISO_{q,r}(N)$. This is achieved by studying the conjugation on the right invariant vector fields h.

For a generic Hopf algebra, with tangent vectors χ_i, we deduce the conjugation on h from the commutation relations between h and a generic element of the Hopf algebra:

$$h_j b = h_j(b) + \kappa'(f^s{}_j)(b_1)\ b_2 h_s$$
$$= \kappa'(\chi_j)(b_1)\ b_2 + \kappa'(f^s{}_j)(b_1)\ b_2 h_s \tag{5.55}$$

We multiply this expression by ${\kappa'}^2(f^j{}_i)(b_0)$ [where we have used the notation $(id \otimes \Delta)\Delta(b) = b_0 \otimes b_1 \otimes b_2$] to obtain

$${\kappa'}^2(f^j{}_i)(b_1)\ h_j b_2 + {\kappa'}^2(\chi_i)(b_1)\,b_2 = bh_i \tag{5.56}$$

Now, using $\psi(b) = \overline{[\kappa'(\psi)]^*(b^*)}$ and then applying $*$ we obtain (here $a = b^*$)

$$h_j^* a = [{\kappa'}^3(\chi_j)]^*(a_1)\ a_2 + [{\kappa'}^3(f^s{}_j)]^*(a_1)\ a_2 h_s \ . \tag{5.57}$$

This last relation implies

$$h_i^* \equiv [h_{\kappa'(\chi_i)}]^* = h_{[\kappa'^3(\chi_i)]^*} \tag{5.58}$$

Notice that $*\circ{\kappa'}^2$ is a well defined conjugation since $(*\circ{\kappa'}^2)^2 = id$.

We now apply formula (5.58), valid for a generic Hopf algebra, to the $*$-conjugation and the right invariant vector fields of this section; we have:

$$[h_{\kappa'(\chi_a)}]^* = -r^N\overline{q}_a d_a^{-1}\mathcal{D}^b{}_a h_{\kappa'(\chi_b)} \tag{5.59}$$
$$[h_{\kappa'(\chi_\bullet)}]^* = -h_{\kappa'(\chi_\bullet)} \ . \tag{5.60}$$

From these last relations and $\partial_C = r^{-1}uh_C$ we finally deduce $(\partial_a)^* = -d_b^{-1}\mathcal{D}^b{}_a r^N\partial_b$ and $(\partial_a)^* = (\partial_\bullet)^* = u - \partial_\bullet$ as in (5.50), (5.51).

5.3 The reduced $SO_{q,r}(N)$-bicovariant algebra generated by x^a, dx^a, ∂_a

Note that the algebra in Table 2 actually contains a subalgebra generated only by x^a, dx^a, ∂_a: indeed $\partial_\bullet$ vanishes when acting on monomials containing only the coordinates x^b, as can be seen from (5.104). This calculus is $ISO_{q,r}(N)$-right covariant because it can also be obtained imposing the conditions $\eta^\bullet = 0$ and $\chi_\bullet = 0$ that are compatible with the right coaction Δ_R and the bimodule structure given by the $f^i{}_j$ functionals.

Table 3 contains the multiparametric orthogonal quantum plane algebra of coordinates, differentials and partial derivatives, together with a consistent conjugation. We emphasize here that this conjugation *does not* require an

Table 1. the $ISO_{q,r}(N)$-bicovariant x^A, ∂_A, dx^A algebra

$$P_A{}^{ab}{}_{cd} x^c x^d = 0 \tag{5.61}$$
$$x^b v = q_b v x^b \;; \quad x^b u = q_b^{-1} u x^b \tag{5.62}$$
$$z = -\frac{1}{(r^{-\frac{N}{2}} + r^{\frac{N}{2}-2})} x^b C_{ba} x^a u \tag{5.63}$$
$$zv = r^2 vz \;; \quad zu = r^{-2} uz \tag{5.64}$$
$$q_a x^a z = z x^a \tag{5.65}$$
$$(x \otimes dx) = (r^2 P_S - P_A - P_0)(dx \otimes x) + P_0 d(x \otimes x) \tag{5.66}$$
$$x^c du = \frac{1}{q_c}(du)x^c - \frac{\lambda}{r}(dx^c)u;$$
$$x^c dv = q_c(dv)x^c + \lambda r(dx^c)v \tag{5.67}$$
$$x^c dz = \frac{1}{q_c}(dz)x^c \tag{5.68}$$
$$u dx^c = \frac{q_c}{r^2}(dx^c)u \tag{5.69}$$
$$u du = r^{-2}(du)u; \quad u dv = r^{-2}(dv)u \tag{5.70}$$
$$u dz = (dz)u \tag{5.71}$$
$$v dx^c = \frac{r^2}{q_c}(dx^c)v \tag{5.72}$$
$$v du = r^2(du)v; \quad v dv = r^2(dv)v \tag{5.73}$$
$$v dz = (dz)v \tag{5.74}$$
$$z dx^c = q_c(dx^c)z \tag{5.75}$$
$$z du = r^{-2}(du)z + (r^{-2} - 1)(dz)u \tag{5.76}$$
$$z dv = r^2(dv)z + (r^2 - 1)(dz)v \tag{5.77}$$
$$z dz = r^{-2}(dz)z \tag{5.78}$$
$$P_S(dx \wedge dx) = 0 \tag{5.79}$$
$$dx^c \wedge du = -\frac{r^2}{q_c} du \wedge dx^c;$$
$$dx^c \wedge dv = -\frac{q_c}{r^2} dv \wedge dx^c \tag{5.80}$$
$$dx^c \wedge dz = -\frac{1}{q_c} dz \wedge dx^c \tag{5.81}$$
$$du \wedge du = dv \wedge dv = 0 \tag{5.82}$$
$$du \wedge dv = -r^{-2} dv \wedge du = 0 \tag{5.83}$$
$$dz \wedge du = -du \wedge dz; \quad dz \wedge dv = -dv \wedge dz \tag{5.84}$$
$$dz \wedge dz = 0 \tag{5.85}$$
$$\partial_c x^b = \delta^b_c I + (r^2 P_S - P_A - P_0)^{be}{}_{cd} x^d \partial_e - (1 - r^2)\delta^b_c v \partial_\bullet \tag{5.86}$$
$$\partial_\bullet x^b = q_b x^b \partial_\bullet - C^{bc} \frac{r^{\frac{N}{2}}(1 - r^2)}{(1 - r^N)} \; z \partial_c \tag{5.87}$$
$$\partial_\circ x^b = q_b^{-1} x^b \partial_\circ - C^{bc} \frac{r^{\frac{N}{2}}(1 - r^2)}{(1 - r^N)} \; v \partial_c \tag{5.88}$$
$$\partial_c v = r^2 q_c^{-1} v \partial_c \tag{5.89}$$
$$\partial_\bullet v = r^2 v \partial_\bullet + I \tag{5.90}$$
$$\partial_\circ v = v \partial_\circ \tag{5.91}$$
$$\partial_c z = q_c z \partial_c \tag{5.92}$$
$$\partial_\bullet z = r^2 z \partial_\bullet \tag{5.93}$$
$$\partial_\circ z = r^{-2} z \partial_\circ + I + (r^2 - 1) v \partial_\bullet \tag{5.94}$$

Table 2. the $SO_{q,r}(N)$ - bicovariant $x^a, v, \partial_a, \partial_\bullet, dx^a, dv$ algebra

$$P_A{}^{ab}{}_{cd} x^c x^d = 0 \tag{5.95}$$
$$x^b v = q_b v x^b \tag{5.96}$$
$$x \otimes dx = r\hat{R}(dx \otimes x) \tag{5.97}$$
$$x^c dv = q_c(dv)x^c + \lambda r(dx^c)v \tag{5.98}$$
$$v dx^c = \frac{r^2}{q_c}(dx^c)v \tag{5.99}$$
$$dx \wedge dx = -r\hat{R} dx \wedge dx \tag{5.100}$$
$$dx^c \wedge dv = -\frac{q_c}{r^2} dv \wedge dx^c \tag{5.101}$$
$$dv \wedge dv = 0 \tag{5.102}$$
$$\partial_c x^b = r\hat{R}^{be}{}_{cd} x^d \partial_e + \delta^b_c[I + (r^2 - 1)v\partial_\bullet] \tag{5.103}$$
$$\partial_\bullet x^b = q_b x^b \partial_\bullet \tag{5.104}$$
$$\partial_\bullet v = r^2 v \partial_\bullet + I \tag{5.105}$$
$$(P_A)^{ab}{}_{cd} \partial_b \partial_a = 0 \tag{5.106}$$
$$\partial_b \partial_\bullet - \frac{q_b}{r^2} \partial_\bullet \partial_b = 0 \tag{5.107}$$

Conjugation:

$$(x^a)^* = \mathcal{D}^a{}_b x^b, \; (dx^a)^* = \mathcal{D}^a{}_b dx^b, \; (\partial_a)^* = -r^N d_a^{-1} \mathcal{D}^b{}_a \partial_b \tag{5.108}$$
$$v^* = v, \; (dv)^* = dv, \; (\partial_\bullet)^* = u - \partial_\bullet \tag{5.109}$$
$$\overline{r} = r^{-1}, \quad \overline{q_a} = \frac{1}{q_a} \text{for } a \neq n, n+1, \quad \overline{q_n} = \frac{1}{q_{n+1}} \tag{5.110}$$

Table 3. the reduced $SO_{q,r}(N)$ -bicovariant x^a, ∂_a, dx^a algebra

$$P_A{}^{ab}{}_{cd} x^c x^d = 0 \tag{5.111}$$
$$x \otimes dx = r\hat{R}(dx \otimes x) \tag{5.112}$$
$$dx \wedge dx = -r\hat{R}(dx \wedge dx) \tag{5.113}$$
$$\partial_c x^b = r\hat{R}^{be}{}_{cd} x^d \partial_e + \delta^b_c I \tag{5.114}$$
$$P_A{}^{ab}{}_{cd} \partial_b \partial_a = 0 \tag{5.115}$$

Conjugation:

$$(x^a)^* = \mathcal{D}^a{}_b x^b, \; (dx^a)^* = \mathcal{D}^a{}_b dx^b, \; (\partial_a)^* = -r^N d_a^{-1} \mathcal{D}^b{}_a \partial_b \tag{5.116}$$

additional scaling operator as in [25]. Thus the algebra in Table 3 can be taken as starting point for a deformed Heisenberg algebra (i.e. a deformed phase-space) .

Real coordinates and hermitean momenta

We note that for the real form $ISO_{q,r}(n+1, n-1)$, the transformation

$$X^a = \frac{1}{\sqrt{2}}(x^a + x^{a'})\,, \quad a \le n \tag{5.117}$$

$$X^{n+1} = \frac{i}{\sqrt{2}}(x^n - x^{n+1}) \tag{5.118}$$

$$X^a = \frac{1}{\sqrt{2}}(x^a - x^{a'})\,, \quad a > n+1 \tag{5.119}$$

defines real coordinates X^a. On this basis the metric becomes $C' = (M^{-1})^T C M^{-1}$ (where M is the transformation matrix $X = Mx$):

$$C' = \frac{1}{2}\begin{pmatrix} r^{\frac{N}{2}-1}+r^{-\frac{N}{2}+1} & 0 & & & 0 & -(r^{\frac{N}{2}-1}-r^{-\frac{N}{2}+1}) \\ 0 & r^{\frac{N}{2}-2}+r^{-\frac{N}{2}+2} & & & -(r^{\frac{N}{2}-2}-r^{-\frac{N}{2}+2}) & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & r^{\frac{N}{2}-2}-r^{-\frac{N}{2}+2} & & & -(r^{\frac{N}{2}-2}+r^{-\frac{N}{2}+2}) & 0 \\ r^{\frac{N}{2}-1}-r^{-\frac{N}{2}+1} & 0 & & & 0 & -(r^{\frac{N}{2}-1}+r^{-\frac{N}{2}+1}) \end{pmatrix} \tag{5.120}$$

and reduces for $r \to 1$ to the usual $SO(n+1, n-1)$ diagonal metric with $n+1$ plus signs and $n-1$ minus signs. Notice that the diagonal elements of C' are real while the off diagonal ones are imaginary; moreover C' is hermitian (and can therefore be diagonalized via a unitary matrix).

As for the coordinates X, it is possible to define antihermitian χ and ∂, and real ω and dx. To define hermitian momenta we first notice that the partial derivatives

$$\tilde{\partial}_a \equiv r^{\frac{N}{2}} d_a^{-\frac{1}{2}} \partial_a \tag{5.121}$$

behave, under the hermitian conjugation $*$, similarly to the coordinates x^a (see (5.50), (5.51)):

$$(\tilde{\partial}_a)^* = -\tilde{\partial}_a \qquad a \neq n, n+1 \tag{5.122}$$

$$(\tilde{\partial}_n)^* = -\tilde{\partial}_{n+1}\ . \tag{5.123}$$

As in (5.117)–(5.119) we then define:

$$P_a = \frac{-i\hbar}{\sqrt{2}}(\tilde{\partial}_a + \tilde{\partial}_{a'})\,, \quad a \le n \tag{5.124}$$

$$P_{n+1} = \frac{-\hbar}{\sqrt{2}}(\tilde{\partial}_n - \tilde{\partial}_{n+1}) \tag{5.125}$$

$$P_a = \frac{-i\hbar}{\sqrt{2}}(\tilde{\partial}_a - \tilde{\partial}_{a'})\,, \quad a > n+1 \tag{5.126}$$

It is easy to see that the P_a are hermitian: ${P_a}^* = P_a$, and that in the classical limit are the momenta conjugated to the coordinates X^a: $P_a(X^b) = -i\hbar\delta^b_a$. In the $r \neq 1$ case we explicitly have (use $d_{a'} = d_a^{-1}\,,\ d_n = d_{n+1} = 1$):

$$P_a(X^a) = -\tfrac{1}{2} i\hbar r^{\frac{N}{2}} (d_a^{\frac{1}{2}} + d_a^{-\frac{1}{2}})$$

$$P_a(X^{a'}) = -\tfrac{1}{2} i\hbar \epsilon_a r^{\frac{N}{2}} (d_a^{\frac{1}{2}} - d_a^{-\frac{1}{2}})$$

where $\epsilon_a = 1$ if $a < n$
and $\epsilon_a = -1$ if $a > n+1$

while the other entries of the $P_a(X^b)$ matrix are zero.

By defining the transformation matrix ${N_a}^b$ as:

$$P_a \equiv -i\hbar {N_a}^b \partial_b \tag{5.127}$$

we find the deformed canonical commutation relations:

$$P_a X^b - r {S^{bc}}_{ad} X^d P_c = -i\hbar E^b_a I \tag{5.128}$$

where

$${S^{bc}}_{ad} = {N_a}^e {M^b}_f {\hat{R}^{fh}}_{eg} {(M^{-1})^g}_d {(N^{-1})_h}^c,$$

$$E^b_a \equiv \frac{i}{\hbar} P_a(X^b) = {N_a}^c {M^b}_c \tag{5.129}$$

Similarly one finds all the remaining commutations of the P, X and dX algebra. Notice that no unitary operator appears on the right-hand side of (5.128). Our conjugation is consistent with (5.128) without the need of the extra operator of [25].

For $n = 2$ the results of this section immediately yield the bicovariant calculus on the quantum Minkowski space, i.e. on the multiparametric orthogonal quantum plane $Fun_{q,r}(ISO(3,1)/SO(3,1))$. The relevant formulas are collected in Appendix A.

A X, dX, P commutations for the D=4 quantum Minkowski space

X real and P hermitean: $X^* = X, P^* = P$

Parameters

Two parameters: r, $q \equiv q_{12}$, with

$$|r| = 1, \quad \frac{q}{r} \in \mathbf{R} \quad \Rightarrow \bar{r} = r^{-1},\ \bar{q} = \frac{q}{r^2} \tag{A.1}$$

Definitions

$$\lambda \equiv r - r^{-1}, \quad \tilde{\lambda} \equiv \frac{r}{q} - \frac{q}{r}, \quad \mu \equiv r + r^{-1}, \quad \tilde{\mu} \equiv \frac{r}{q} + \frac{q}{r} \tag{A.2}$$

XX commutations

$$X^2X^1 = \frac{1}{\mu^2+\tilde{\lambda}^2}[\mu\tilde{\mu}X^1X^2 + \mu\lambda X^2X^4 + i\tilde{\lambda}\tilde{\mu}X^3X^4 - i\tilde{\lambda}\lambda X^1X^3]$$
$$X^3X^1 = \frac{1}{\mu^2+\tilde{\lambda}^2}[\mu\tilde{\mu}X^1X^3 + \mu\lambda X^3X^4 - i\tilde{\lambda}\tilde{\mu}X^2X^4 + i\tilde{\lambda}\lambda X^1X^2]$$
$$X^4X^1 = X^1X^4 + \frac{\lambda}{2}(X^2X^2 + X^3X^3)$$
$$X^3X^2 = X^2X^3$$
$$X^4X^2 = \frac{1}{\mu^2+\tilde{\lambda}^2}[\mu\tilde{\mu}X^2X^4 - \mu\lambda X^1X^2 - i\tilde{\lambda}\tilde{\mu}X^1X^3 - i\tilde{\lambda}\lambda X^3X^4]$$
$$X^4X^3 = \frac{1}{\mu^2+\tilde{\lambda}^2}[\mu\tilde{\mu}X^3X^4 - \mu\lambda X^1X^3 + i\tilde{\lambda}\tilde{\mu}X^1X^2 + i\tilde{\lambda}\lambda X^2X^4]$$

X dX commutations

$$X^1dX^1 = \frac{1}{4}(r^{-2}+3r^2)(dX^1)X^1 - \frac{\lambda^2}{4}[(dX^4)X^1 - (dX^1)X^4] - \frac{\lambda}{2}[(dX^2)X^2 + (dX^3)X^3] + \frac{1}{4}(r^2 - r^{-2})(dX^4)X^4$$
$$X^1dX^2 = \frac{1}{2}r\tilde{\mu}(dX^2)X^1 + \frac{r\lambda}{2}d[X^1 - X^4]X^2 - \frac{i}{2}r\tilde{\lambda}(dX^3)X^4$$
$$X^1dX^3 = \frac{1}{2}r\tilde{\mu}(dX^3)X^1 + \frac{r\lambda}{2}d[X^1 - X^4]X^3 + \frac{i}{2}r\tilde{\lambda}(dX^2)X^4$$
$$X^1dX^4 = [\frac{1}{4}(r^2 - r^{-2}) + 1](dX^4)X^1 + \frac{\lambda^2}{4}[(dX^1)X^1 - (dX^4)X^4] - \frac{\lambda}{2}[(dX^2)X^2 + (dX^3)X^3] + [\frac{1}{4}(3r^2 + r^{-2}) - 1](dX^1)X^4$$
$$X^2dX^1 = \frac{1}{2}r\tilde{\mu}(dX^1)X^2 + \frac{r\lambda}{2}dX^2(X^1 + X^4) + \frac{i}{2}r\tilde{\lambda}(dX^4)X^3$$
$$X^2dX^2 = \frac{r\mu}{2}(dX^2)X^2 - \frac{r\lambda}{2}(dX^3)X^3 - \frac{\lambda}{2}d[X^1 - X^4](X^1 + X^4)$$
$$X^2dX^3 = \frac{r\mu}{2}(dX^3)X^2 + \frac{r\lambda}{2}(dX^2)X^3$$
$$X^2dX^4 = \frac{1}{2}r\tilde{\mu}(dX^4)X^2 + \frac{r\lambda}{2}dX^2(X^1 + X^4) + \frac{i}{2}r\tilde{\lambda}(dX^1)X^3$$
$$X^3dX^1 = \frac{1}{2}r\tilde{\mu}(dX^1)X^3 + \frac{r\lambda}{2}dX^3(X^1 + X^4) - \frac{i}{2}r\tilde{\lambda}(dX^4)X^2$$
$$X^3dX^2 = \frac{r\mu}{2}(dX^2)X^3 + \frac{r\lambda}{2}(dX^3)X^2$$
$$X^3dX^3 = \frac{r\mu}{2}(dX^3)X^3 - \frac{r\lambda}{2}(dX^2)X^2 - \frac{\lambda}{2}d[X^1 - X^4](X^1 + X^4)$$
$$X^3dX^4 = \frac{1}{2}r\tilde{\mu}(dX^4)X^3 + \frac{r\lambda}{2}dX^3(X^1 + X^4) - \frac{i}{2}r\tilde{\lambda}(dX^1)X^2$$
$$X^4dX^1 = [\frac{1}{4}(r^2 - r^{-2}) + 1](dX^1)X^4 - \frac{\lambda^2}{4}[(dX^1)X^1 - (dX^4)X^4] + \frac{\lambda}{2}[(dX^2)X^2 + (dX^3)X^3] + [\frac{1}{4}(3r^2 + r^{-2}) - 1](dX^4)X^1$$
$$X^4dX^2 = \frac{1}{2}r\tilde{\mu}(dX^2)X^4 - \frac{r\lambda}{2}d[X^1 - X^4]X^2 - \frac{i}{2}r\tilde{\lambda}(dX^3)X^1$$
$$X^4dX^3 = \frac{1}{2}r\tilde{\mu}(dX^3)X^4 - \frac{r\lambda}{2}d[X^1 - X^4]X^3 + \frac{i}{2}r\tilde{\lambda}(dX^2)X^1$$
$$X^4dX^4 = \frac{1}{4}(r^{-2}+3r^2)(dX^4)X^4 + \frac{\lambda^2}{4}[(dX^4)X^1 - (dX^1)X^4] + \frac{\lambda}{2}[(dX^2)X^2 + (dX^3)X^3] + \frac{1}{4}(r^2 - r^{-2})(dX^1)X^1$$

dX dX commutations

(Products between dX are exterior (wedge) products)

$$dX^1dX^1 = 0$$
$$dX^1dX^2 = \frac{1}{\mu^2-\tilde{\lambda}^2}[-\mu\tilde{\mu}dX^2dX^1 + \mu\lambda dX^4dX^2 - i\tilde{\lambda}\tilde{\mu}dX^4dX^3 - i\lambda\tilde{\lambda}dX^3dX^1]$$
$$dX^1dX^3 = \frac{1}{\mu^2-\tilde{\lambda}^2}[-\mu\tilde{\mu}dX^3dX^1 + \mu\lambda dX^4dX^3$$

$$+i\tilde{\lambda}\tilde{\mu}dX^4dX^2 + i\lambda\tilde{\lambda}dX^2dX^1]$$
$$dX^1dX^4 = -dX^4dX^1$$
$$dX^2dX^2 = -\frac{\lambda}{2}dX^4dX^1$$
$$dX^2dX^3 = -dX^3dX^2$$
$$dX^2dX^4 = \frac{1}{\mu^2-\tilde{\lambda}^2}[-\mu\tilde{\mu}dX^4dX^2 - \mu\lambda dX^2dX^1 + i\tilde{\lambda}\tilde{\mu}dX^3dX^1 - i\lambda\tilde{\lambda}dX^4dX^3]$$
$$dX^3dX^3 = -\frac{\lambda}{2}dX^4dX^1$$
$$dX^3dX^4 = \frac{1}{\mu^2-\tilde{\lambda}^2}[-\mu\tilde{\mu}dX^4dX^3 - \mu\lambda dX^3dX^1 - i\tilde{\lambda}\tilde{\mu}dX^2dX^1 + i\lambda\tilde{\lambda}dX^4dX^2]$$
$$dX^4dX^4 = 0$$

P X commutations

Defining

$$A \equiv -\frac{r\lambda}{4}[(X^1-X^4)(P_1-P_4) + r^2(X^1+X^4)(P_1+P_4) + 2r(X^2P_2+X^3P_3)]$$
$$B \equiv -\frac{r\lambda}{4}[(X^1-X^4)(P_1-P_4) - r^2(X^1+X^4)(P_1+P_4) - 2r(X^2P_2+X^3P_3)]$$

the commutations are:

$$P_1X^1 - X^1P_1 + A = -\frac{1}{2}i\hbar r^2\mu$$
$$P_1X^2 - \frac{r\tilde{\mu}}{2}X^2P_1 - \frac{r}{2}[-\lambda(X^1+X^4)P_2 + i\tilde{\lambda}X^3P_4] = 0$$
$$P_1X^3 - \frac{r\tilde{\mu}}{2}X^3P_1 - \frac{r}{2}[-\lambda(X^1+X^4)P_3 - i\tilde{\lambda}X^2P_4] = 0$$
$$P_1X^4 - r^2X^4P_1 + B - r\lambda X^1P_4 = \frac{1}{2}i\hbar r^2\lambda$$
$$P_2X^1 - \frac{r\tilde{\mu}}{2}X^1P_2 - \frac{r}{2}[-\lambda X^2(P_1+P_4) + i\tilde{\lambda}X^4P_3] = 0$$
$$P_2X^2 - \frac{r\mu}{2}X^2P_2 - \frac{r\lambda}{2}[X^3P_3 + r(X^1+X^4)(P_1+P_4)] = -i\hbar r^2$$
$$P_2X^3 - \frac{r\mu}{2}X^3P_2 + \frac{r\lambda}{2}X^2P_3 = 0$$
$$P_2X^4 - \frac{r\tilde{\mu}}{2}X^4P_2 - \frac{r}{2}[\lambda X^2(P_1+P_4) + i\tilde{\lambda}X^1P_3] = 0$$
$$P_3X^1 - \frac{r\tilde{\mu}}{2}X^1P_3 - \frac{r}{2}[-\lambda X^3(P_1+P_4) - i\tilde{\lambda}X^4P_2] = 0$$
$$P_3X^2 - \frac{r\mu}{2}X^2P_3 + \frac{r\lambda}{2}X^3P_2 = 0$$
$$P_3X^3 - \frac{r\mu}{2}X^3P_3 - \frac{r\lambda}{2}[X^2P_2 + r(X^1+X^4)(P_1+P_4)] = -i\hbar r^2$$
$$P_3X^4 - \frac{r\tilde{\mu}}{2}X^4P_3 - \frac{r}{2}[\lambda X^3(P_1+P_4) - i\tilde{\lambda}X^1P_2] = 0$$
$$P_4X^1 - r^2X^1P_4 + B - r\lambda X^4P_1 = \frac{1}{2}i\hbar r^2\lambda$$
$$P_4X^2 - \frac{r\tilde{\mu}}{2}X^2P_4 - \frac{r}{2}[\lambda(X^1+X^4)P_2 + i\tilde{\lambda}X^3P_1] = 0$$
$$P_4X^3 - \frac{r\tilde{\mu}}{2}X^3P_4 - \frac{r}{2}[\lambda(X^1+X^4)P_3 - i\tilde{\lambda}X^2P_1] = 0$$
$$P_4X^4 - X^4P_4 + A = -\frac{1}{2}i\hbar r^2\mu$$

P P commutations

$$P^2P^1 = \frac{1}{\mu^2+\tilde{\lambda}^2}[\mu\tilde{\mu}P^1P^2 - \mu\lambda P^2P_4 - i\tilde{\lambda}\tilde{\mu}P_3P_4 - i\tilde{\lambda}\lambda P_1P_3]$$
$$P^3P^1 = \frac{1}{\mu^2+\tilde{\lambda}^2}[\mu\tilde{\mu}P^1P^3 - \mu\lambda P^3P_4 + i\tilde{\lambda}\tilde{\mu}P_2P_4 + i\tilde{\lambda}\lambda P_1P_2]$$
$$P^4P^1 = P^1P^4 - \frac{\lambda}{2}(P_2P_2 + P_3P_3)$$
$$P^3P^2 = P^2P^3$$
$$P^4P^2 = \frac{1}{\mu^2+\tilde{\lambda}^2}[\mu\tilde{\mu}P^2P^4 + \mu\lambda P^1P_2 + i\tilde{\lambda}\tilde{\mu}P_1P_3 - i\tilde{\lambda}\lambda P_3P_4]$$
$$P^4P^3 = \frac{1}{\mu^2+\tilde{\lambda}^2}[\mu\tilde{\mu}P^3P^4 + \mu\lambda P^1P_3 - i\tilde{\lambda}\tilde{\mu}P_1P_2 + i\tilde{\lambda}\lambda P_2P_4]$$

B R matrix of orthogonal q-groups: properties

Let $\hat{R}$ be the matrix defined by $\hat{R}^{ab}{}_{cd} \equiv R^{ba}{}_{cd}$.

Characteristic equation and projector decomposition:

$$(\hat{R}-rI)(\hat{R}+r^{-1}I)(\hat{R}-r^{1-N}I) = 0 \tag{B.1}$$
$$\hat{R}-\hat{R}^{-1} = (r-r^{-1})(I-K) \tag{B.2}$$
$$\hat{R} = rP_S - r^{-1}P_A + r^{1-N}P_0 \tag{B.3}$$

with

$$\begin{array}{l} P_S = \frac{1}{r+r^{-1}}[\hat{R}+r^{-1}I-(r^{-1}+r^{1-N})P_0] \\ P_A = \frac{1}{r+r^{-1}}[-\hat{R}+rI-(r-r^{1-N})P_0] \\ P_0 = Q_N(r)K \\ Q_N(r) \equiv (C_{ab}C^{ab})^{-1} = \frac{1-r^{-2}}{(1-r^{-N})(1+r^{N-2})}\ , \\ K^{ab}{}_{cd} = C^{ab}C_{cd} \\ I = P_S+P_A+P_0 \end{array} \tag{B.4}$$

Other properties involving the q-metric:

$$C_{ab}\hat{R}^{bc}{}_{de} = (\hat{R}^{-1})^{cf}{}_{ad}C_{fe}, \quad \hat{R}^{bc}{}_{de}C^{ea} = C^{bf}(\hat{R}^{-1})^{ca}{}_{fd} \tag{B.5}$$
$$C_{ab}\hat{R}^{ab}{}_{cd} = r^{1-N}C_{cd}, \quad C^{cd}\hat{R}^{ab}{}_{cd} = r^{1-N}C^{ab} \tag{B.6}$$

The identities (B.5) hold also for $\hat{R} \to \hat{R}^{-1}$.

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